The Fundamental Theorem of Calculus

There are two parts to the Fundamental Theorem of Calculus (FTC).

Part 1: \[ \frac{d}{dx} \int_0^x f(t) \, dt = f(x) \]

Part 2: \[ \int_a^b f(x) \, dx = F(b) - F(a), \text{ where } F'(x) = f(x) \]

The easiest way to understand these equations is to introduce what we call an area function. Given a function \( f(x) \), define the area function of \( f(x) \) as

\[ A(x) = \int_0^x f(t) \, dt \] (1)

The area function \( A(x) \) gives the area under the curve \( f(x) \) from 0 to \( x \). Note that the integral is written using the variable \( t \). This is so that the variable \( x \) can be used for the upper limit of integration. See Figure 1 and the link to the graph for an example of an area function.

**Figure 1 - Area Function of \( f(x) = x \)**

[https://www.desmos.com/calculator/fbkfqoyzjd](https://www.desmos.com/calculator/fbkfqoyzjd)

**Part 1**

Using equation (1), the left hand side of Part 1 can be rewritten as

\[ \frac{d}{dx} \int_0^x f(t) \, dt = \frac{d}{dx} A(x) = A'(x) \]

so that Part 1 becomes

\[ \text{Part 1: } A'(x) = f(x) \] (2)

which says that the derivative of the area function of \( f(x) \) is just \( f(x) \). In other words, the area function \( A(x) \) is an anti-derivative of \( f(x) \).
Proving equation (2) is the hard part which we won’t do here*. But equation (2) is demonstrated in Figure 1 where, by elementary geometry, the area function for \( f(x) = x \) is seen to be

\[ A(x) = \frac{1}{2} x^2. \]

**Part 2**

Part 2 of the FTC shows how you can use any anti-derivative of \( f(x) \) to compute the area under the curve \( f(x) \). This result follows readily from Part 1. Let \( F(x) \) be any anti-derivative of \( f(x) \) so that \( F'(x) = f(x) \). Part 1 says that \( A(x) \) is also an anti-derivative of \( f(x) \). Since both \( F(x) \) and \( A(x) \) are anti-derivatives of \( f(x) \), they must differ by a constant. In other words, for some constant \( C \), we have

\[ A(x) = F(x) + C \quad (3) \]

Now, for \( 0 \leq a \leq b \), it's clear that

\[ \int_a^b f(x) \, dx = \int_0^a f(x) \, dx + \int_a^b f(x) \, dx \quad (4) \]

Combining equations (1), (3), and (4), gives us Part 2,

\[ \int_a^b f(x) \, dx = \int_a^b f(x) \, dx - \int_0^a f(x) \, dx
\]
\[ = A(b) - A(a)
\]
\[ = (F(b) + C) - (F(a) + C)
\]
\[ = F(b) - F(a) \]

**FTC Restated**

Now we can restate the FTC in terms of the area function and anti-derivatives.

**Fundamental Theorem of Calculus**

Let \( f(x) \) be a function and let \( A(x) = \int_0^x f(t) \, dt \) be its area function. Then

Part 1: \( A'(x) = f(x) \). \( A \) is an anti-derivate of \( f \).

Part 2: \( \int_a^b f(x) \, dx = F(b) - F(a) \), for any anti-derivative \( F \) of \( f \).

*Proving equation (2) is intuitive using infinitesimals. Let \( \delta \) be any non-zero infinitesimal. Then,

\[ A'(x) \approx \frac{A(x+\delta) - A(x)}{\delta} = \frac{\int_0^{x+\delta} f(t) \, dt - \int_0^{x} f(t) \, dt}{\delta} = \frac{\int_x^{x+\delta} f(t) \, dt}{\delta} \approx \frac{f(x)\delta}{\delta} = f(x) \]