Differential Equations Definitions

- A **differential equation** is an equation with derivatives.
  - The solution of a differential equation is a function.
- There are two major types of differential equations.
  - **ODE**, Ordinary Differential Equation
    - An ODE is a diff eq with just one independent variable and only ordinary derivatives.
    - In this course, we study only ODE’s. From now on, when we say “differential equation” or “diff eq”, we mean an ODE.
  - **PDE**, Partial Differential Equation
    - An PDE is a diff eq with more than one independent variable and has partial derivatives.
- The **order of a differential equation** is the order of the highest order derivative in the differential equation.
- The **normal form** of a differential equation has the highest order derivative by itself on the left hand side of the equation.

- A **separable differential equation** is a first order diff eq of the form
  \[ y' = g(x) \cdot h(y) \]
- A **linear differential equation** is a diff eq of the form
  \[ a_n(x) y^{(n)} + \ldots + a_2(x) y'' + a_1(x) y' + a_0(x) y = g(x) \]
  - Each term on the left hand side is \( y \) or one of its derivatives times a **coefficient function** of \( x \). The right hand side can be any function of \( x \).
  - In linear models, the left hand side is the physical system, \( g(x) \) is the **forcing function**, and the solution is the **response** of the system.
  - \( a_n(x) \) is called the **leading coefficient function**.
  - The zeros of the leading coefficient function \( a_n(x) \) are called **singular points**. Things go haywire at singular points. Singular points won't be in the domain of any solution.
  - A linear diff eq is called "linear" because the left hand side constitutes a “linear operator” of \( y \).

- The **solution of differential equation** is a function along with an **interval of definition** on which the function solves the diff eq.
  - The solution must be continuous and differentiable on the interval of definition.
  - Strictly speaking, a complete solution must specify an interval of definition along with the function. In practice, the interval of definition is often overlooked or not mentioned.
- A **response curve** is the solution of a diff eq in the context of a model of a physical system.
  - Because of this, we can say safely that the solution of a diff eq must be the possible response curve of some physical system. Hence, the solution of a diff eq must be a
continuous and differentiable function on a single interval. (Anything we encounter in real life is continuous and differentiable and doesn’t jump time intervals.)

- The function \( y = 0 \) is called the **trivial solution** when it solves the diff eq.
  - For example: The diff eq \( y' + y = 0 \) has the trivial solution since the function \( y = 0 \) solves the diff eq.

- An **explicit function** is a function defined explicitly by a formula.
- An **implicit function** is a function defined implicitly by an \( x,y \) equation. The \( x,y \) equation itself is called the **implicit equation** or the **implicit curve**.
  - Every \( x,y \) equation defines a **relation** which graphs as a curve in the \( x,y \) plane.
  - If you can enclose in a rectangle a piece of the curve that looks like a function, then that function is “implicitly defined” by the relation.
  - For example, parts of the implicit curve \( x - y^2 = 0 \) (sideways parabola) implicitly define functions of \( x \), including the explicit functions \( y_1 = \sqrt{x} \) and \( y_2 = -\sqrt{x} \).

- An **explicit solution** is a solution with an explicit function.
- An **implicit solution** is a solution with an implicit function.
  - The relation \( x^2 + y^2 = 1 \) is an implicit solution of the diff eq \( y' = -x/y \) because all the functions \( y(x) \) implicitly defined by \( x^2 + y^2 = 1 \) solve the diff eq.
  - In most cases we state an implicit solution without stating an interval of definition.

- An **\( n \)-parameter family of solutions** has \( n \) parameters (arbitrary constants) usually labeled \( c_1, c_2, \ldots, c_n \). For example, \( y = c_1 \cos t + c_2 \sin t \) is a two parameter family of functions.
- A **particular solution** has no parameters.
  - You get particular solution from an \( n \)-parameter family by setting the values of the parameters \( c_1, c_2, \ldots, c_n \). A particular solution is a particular member of the family of solutions.
  - For example, \( x(t) = c_1 \cos t + c_2 \sin t \) is the two parameter family of solutions of the diff eq \( x'' + x = 0 \). The function \( x(t) = 2 \cos t - 3 \sin t \) is a particular solution from that two parameter family where \( c_1 = 2 \) and \( c_2 = -3 \).
- A **singular solution** is a solution that's not the member of a parametrized family of solutions.
  - For example, \( y = \left( \frac{1}{4} x^2 + c_1 \right)^2 \) is a one parameter family of solutions of the diff eq \( y' = x \sqrt{y} \) (verify). Also \( y = 0 \) solves the diff eq (verify). But \( y = 0 \) is not a member of the family (verify: there's no value of \( c_1 \) that gives you \( y = 0 \)). So, \( y = 0 \) is a singular solution.
  - Singular solutions often "drop out" of calculations when dividing. For example, when solving the previous diff eq \( y' = x \sqrt{y} \) by separation of variables, we lose \( y = 0 \) when we divide by \( \sqrt{y} \). (Verify by solving the diff eq.)
A **general solution** is a family of functions that represents all possible solutions to a diff eq.
- If a diff eq has a general solution, it will have no singular solutions.
- Solving a linear diff eqs yields a general solution.
- Solving separable diff eqs often does not yield a general solution.

**Initial Value Problem (IVP)**
- An **nth order Initial Value Problem** is an n-th order diff eq along with n **initial conditions**.
- The n initial conditions are on y and its first \( n - 1 \) derivatives:
  \[ y(x_0), y'(x_0), y''(x_0), \ldots, y^{(n-1)}(x_0). \]
- The initial conditions are all evaluated at the same **initial point** \( x_0 \).
- Solving an IVP happens in two stages
  1. Solve the diff eq to get the \( n \)-parameter family of solutions.
  2. Apply the initial conditions to find the particular solution.
- The solution to an \( n \)th order IVP is a particular solution that belongs to the \( n \)-parameter family of solutions.
- Solutions to **Linear IVPs** will often have these features.
  - **transient terms** - any term whose value diminishes as \( t \to \infty \).
  - **steady state terms** - any term that’s not a transient term.
  - The **transient solution** is all the transient terms of the solution.
  - The **steady state solution** is all the steady state terms of the solution.
  - For example, the linear IVP \( y' + 2y = 8 \), \( y(0) = 1 \) has the particular solution \( y = 4 - 3e^{-2t} \) (verify). The solution response curve has a transient term of \( -3e^{-2t} \) and a steady state of \( y = 4 \).

![Response Curve of the IVP](image)