Chapter 7

A Foreign Exchange Model

7.1 The assets and the risk-neutral measures

In this section we consider an $N$-period binomial model, but with two currencies. In particular, there is a *domestic interest rate* $r \geq 0$ and a *foreign interest rate* $r^f \geq 0$. The domestic interest rate leads to a *domestic money market account* whose price at each time $n$ is

$$M_n = (1 + r)^n, \quad n = 0, 1, \ldots, N. \quad (7.1.1)$$

The foreign interest rate leads to a *foreign money market account* whose price at each time $n$ is

$$M_n^f = (1 + r^f)^n, \quad n = 0, 1, \ldots, N. \quad (7.1.2)$$

The domestic money market account is denominated in domestic currency and the foreign money market account is denominated in foreign currency.

We also have an *exchange rate process* $Q_n$, where $Q_n$ is the price of one unit of foreign currency in units of domestic currency at time $n$. The initial exchange rate $Q_0$ is assumed to be positive, and $Q_{n+1}$ is defined in terms of $Q_n$ by the recursive equation

$$Q_{n+1}(\omega_1 \ldots \omega_n \omega_{n+1}) = \begin{cases} 
  u_Q Q_n(\omega_1 \ldots \omega_n) & \text{if } \omega_{n+1} = H, \\
  d_Q Q_n(\omega_1 \ldots \omega_n) & \text{if } \omega_{n+1} = T. 
\end{cases} \quad (7.1.3)$$

The *up factor* $u_Q$ and *down factor* $d_Q$ for the exchange rate satisfy

$$0 < d_Q < \frac{1 + r}{1 + r^f} < u_Q. \quad (7.1.4)$$

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Although it is not really necessary for our analysis, we shall assume that \( u_Q > 1 \) and \( d_Q < 1 \) so the exchange rate really does increase when a coin toss results in \( H \) and really does decrease when a coin toss results in \( T \).

In the domestic currency there are two assets in which an agent can invest. One is the domestic money market account, and the other is the foreign money market account converted to domestic currency. The price of the latter asset at time \( n \) is

\[
S_n = M_n^f Q_n = (1 + r^f)^n Q_n. \tag{7.1.5}
\]

Whereas the exchange rate has an up factor \( u_Q \) and a down factor \( d_Q \), the asset \( S \) has an up and down factors

\[
u = (1 + r^f)u_Q, \quad d = (1 + r^f)d_Q. \tag{7.1.6}
\]

Even if \( d_Q < 1 \), it is not necessarily the case that \( d < 1 \); thus, the term “down factor” should not be taken literally. We are only assured that \( d < u \) because of (7.1.4), so a coin toss of a tail results in a worse performance for \( S \) than a coin toss of a head, although this worse performance may not actually be a decrease in price.

From (7.1.4) and (7.1.5) we see that

\[
0 < d < 1 + r < u, \tag{7.1.7}
\]

which is the no-arbitrage condition for the binomial model. In particular, we may define risk-neutral probabilities

\[
\tilde{p} = \frac{1 + r - d}{u - d}, \quad \tilde{q} = \frac{u - 1 - r}{u - d}, \tag{7.1.8}
\]

and define the domestic risk-neutral measure \( \tilde{P} \) to correspond to independent coin tosses with probability \( \tilde{p} \) for \( H \) on each toss and probability \( \tilde{q} \) for \( T \) on each toss. Under \( \tilde{P} \), the discounted process \( S \) is a martingale:

\[
\tilde{E}_n \left[ \frac{S_{n+1}}{M_{n+1}} \right] = \frac{S_n}{M_n}, \quad n = 0, 1, \ldots, N - 1, \tag{7.1.9}
\]

Furthermore, \( \tilde{P} \) is the only probability measure under which discounted \( S \) is a martingale. Recalling the definition (7.1.5) of \( S \), we see that

\[
\tilde{E}_n \left[ \frac{M_n^f Q_{n+1}}{M_{n+1}} \right] = \frac{M_n^f Q_n}{M_n}, \quad n = 0, 1, \ldots, N - 1. \tag{7.1.10}
\]
In other words,
\[
\frac{M_n^f Q_n}{M_n} = \left( \frac{1 + r^f}{1 + r} \right)^n Q_n, \quad n = 0, 1, \ldots, N,
\] (7.1.11)
is a martingale under \( \tilde{P} \).

In the foreign currency there are also two assets in which an agent can invest. One is the foreign money market account, and the other is the domestic money market account converted to foreign currency. The price of the latter asset at time \( n \) is
\[
W_n = \frac{M_n}{Q_n} = \left( \frac{1 + r}{Q} \right)^n. \tag{7.1.12}
\]

When a coin toss results in a head, the exchange rate \( Q \) increases by the factor \( u_Q \), so the reciprocal exchange rate decreases by the factor \( \frac{1}{u_Q} \). Therefore, \( W \) “decreases” by the factor \( d_W = \frac{1 + r}{u_Q} \). When the coin toss results in a tail, the exchange rate \( Q \) decreases by the factor \( d_Q \), so the reciprocal exchange rate increases by the factor \( \frac{1}{d_Q} \). Therefore, \( W \) increases by the factor \( u_W = \frac{1 + r}{d_Q} \). We do not really know that \( d_W \) is less than one, but we can see that \( d_W < u_W \). Moreover, from (1.4), we see that with the “down” and up factors for \( W \) defined by the above formulas, which we repeat here,
\[
u_W = \frac{1 + r}{d_Q}, \quad d_W = \frac{1 + r}{u_Q}, \tag{7.1.13}
\]
satisfy the no-arbitrage condition in the foreign currency
\[
0 < d_W < 1 + r^f < u_W. \tag{7.1.14}
\]

Because a \( T \) results in an up move of \( W \) and an \( H \) results in a down move, which is opposite to the effect of \( H \) and \( T \) on \( S \), we define the foreign risk-neutral probabilities opposite to the formula (7.1.8) for the domestic risk-neutral probabilities. In particular, we define
\[
q^f = \frac{1 + r^f - d_W}{u_W - d_W}, \quad p^f = \frac{u_W - 1 - r^f}{u_W - d_W}, \tag{7.1.15}
\]
and define the foreign risk-neutral measure \( \tilde{P}^f \) to correspond to independent coin tosses with probability \( \tilde{p}^f \) for \( H \) (a “down” move of \( W \)) on each toss and probability \( \tilde{q}^f \) for \( T \) (an up move of \( W \)) on each toss. Under \( \tilde{P}^f \),
the discounted (using the foreign money market account) process $W$ is a martingale,

$$\tilde{E}_f^n \left[ \frac{W_{n+1}}{M_{n+1}^f} \right] = \frac{W_n}{M_n^f}, \quad n = 0, 1, \ldots, N - 1, \quad (7.1.16)$$

as we now show. If the first $n$ coin tosses result in $w = \omega_1 \ldots \omega_n$, then

$$\tilde{E}_f^n \left[ \frac{W_{n+1}}{M_{n+1}^f} \right] (w)$$

$$= \tilde{p}^f \frac{W_{n+1}(wH)}{M_{n+1}^f} + \tilde{q}^f \frac{W_{n+1}(wT)}{M_{n+1}^f}$$

$$= \frac{uW - 1 - r_f}{uW - dW} \cdot \frac{dW W_n(w)}{(1 + r_f)M_n^f} + \frac{1 + r_f - dW}{uW - dW} \cdot \frac{uW W_n(w)}{(1 + r_f)M_n^f}$$

$$= \frac{(uW - 1 - r_f)dW + (1 + r_f - dW)uW}{(uW - dW)(1 + r_f)} \cdot \frac{W_n(w)}{M_n^f}$$

$$= \frac{uW dW - dW - r_f dW + uW + r_f uW - dW uW}{(uW - dW)(1 + r_f)} \cdot \frac{W_n(w)}{M_n^f}$$

$$= \frac{W_n(w)}{M_n^f}.$$ 

Since $w = \omega_1 \ldots \omega_n$ is an arbitrary sequence of coin tosses, (7.1.16) holds regardless of the outcome of the coin tosses. Recalling the definition (7.1.12) of $W$, we see that

$$\tilde{E}_f^n \left[ \frac{M_{n+1}}{M_{n+1}^f Q_{n+1}} \right] = \frac{M_n}{M_n^f Q_n}, \quad n = 0, 1, \ldots, N - 1. \quad (7.1.17)$$

In other words,

$$\frac{M_n}{M_n^f Q_n} = \left( \frac{1 + r}{1 + r_f} \right)^n \frac{1}{Q_n}, \quad n = 0, 1, \ldots, N, \quad (7.1.18)$$

is a martingale under $\tilde{P}^f$. Furthermore, $\tilde{P}^f$ is the only probability measure under which this process is a martingale.
7.2. **EXAMPLE**

\[
Q_2(TH) = Q_2(HT) = 2
\]

\[
Q_0 = 2
\]

\[
Q_1(T) = \frac{4}{3}
\]

\[
Q_1(H) = 3
\]

\[
Q_2(TT) = \frac{8}{5}
\]

\[
Q_2(HT) = Q_2(TH) = 2
\]

\[
Q_2(HH) = \frac{9}{2}
\]

Figure 7.2.1: A two-period model.

### 7.2 Example

We consider a two-period example in which the initial exchange rate is \(Q_0 = 2\) and the up and down factors for the exchange rate are

\[
u_Q = \frac{3}{2}, \quad d_Q = \frac{2}{3},
\]  

(7.2.1)

The resulting exchange rate process is shown in Figure 7.2.1.

We take the domestic and foreign interest rates to be

\[
r = \frac{1}{4}, \quad r_f = \frac{1}{3},
\]  

(7.2.2)

so that

\[
1 + r = \frac{5}{4}, \quad \frac{1}{1 + r} = \frac{4}{5}, \quad 1 + r_f = \frac{4}{3}, \quad \frac{1}{1 + r_f} = \frac{3}{4},
\]  

(7.2.3)

Note that

\[
d_Q = \frac{2}{3} < \frac{1 + r}{1 + r_f} = \frac{15}{16} < u_Q = \frac{3}{2},
\]  

(7.2.4)

as required by (7.1.4), so the no-arbitrage conditions (7.1.8) and (7.1.14) are satisfied.

The up and down factors for \(S\), the foreign money market denominated in domestic currency, are given by (7.1.6). In this example these are

\[
u = (1 + r_f)u_Q = \frac{4}{3} \cdot \frac{3}{2} = 2, \quad d = (1 + r_f)d_Q = \frac{4}{3} \cdot \frac{2}{3} = \frac{8}{9},
\]  

(7.2.5)
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The process $S$ is shown in Figure 7.2.2. The risk-neutral probabilities, given by (7.1.8), are

$$
\tilde{p} = \frac{1 - r - d}{u - d} = 1 + \frac{1}{3} - \frac{8}{9} = \frac{13}{40}, \quad \tilde{q} = 1 - \tilde{p} = \frac{27}{40}.
$$

(7.2.6)

The up and down factors for the domestic money market denominated in foreign currency, given by (7.1.13), are

$$
d_W = \frac{1 + r}{uQ} = \frac{5}{4} \cdot \frac{2}{3} = \frac{5}{8}, \quad u_W = \frac{1 + r}{dQ} = \frac{5}{4} \cdot \frac{3}{2} = \frac{15}{8}.
$$

(7.2.7)

The price process for the domestic money market denominated in foreign currency, given by (7.1.12), is shown in Figure 7.2.3. The risk-neutral probabilities under the domestic currency, given by (7.1.15), are

$$
\tilde{q}^f = \frac{1 + r^f - d_W}{u_W - d_W} = \frac{1 + \frac{1}{3} - \frac{5}{8}}{\frac{15}{8} - \frac{5}{6}} = \frac{12}{25}, \quad \tilde{p}^f = 1 - \tilde{q}^f = \frac{13}{25}.
$$

(7.2.8)

One can verify that $\frac{W}{M^f}$ is a martingale under $\tilde{P}^f$, the probability measure corresponding to independent coin tosses with probability $\tilde{p}^f$ for each $H$ and $\tilde{q}^f$ for each tail. For example,

$$
\tilde{E}^f \left[ \frac{W_2}{M_2^f} \right] (H) = \tilde{p}^f \frac{W_2(HH)}{M_2^f(HH)} + \tilde{q}^f \frac{W_2(HT)}{M_2^f(HT)} = \frac{13}{25} \cdot \frac{25}{9} + \frac{12}{25} \cdot \frac{25}{32} = \frac{9}{16} \cdot \frac{5}{9} = \frac{5}{16} = \frac{W_1(H)}{M_1^f}.
$$
7.3. RELATIONS BETWEEN RISK-NEUTRAL MEASURES

To understand the relationship between the risk-neutral measure \( \tilde{\mathbb{P}} \) for the domestic currency and the risk-neutral measure \( \tilde{\mathbb{P}}^f \) for the foreign currency, we shall rely on the following theorem.

**Theorem 7.3.1 (Quotient of martingales)** In an \( N \)-period binomial model, suppose that \( X_n \) and \( Z_n \), \( n = 0, 1, \ldots, N \) are martingales under a probability measure \( \mathbb{P} \) such that \( \mathbb{P}(\omega_1 \ldots \omega_N) > 0 \) for every \( \omega_1 \ldots \omega_N \). Suppose further that \( Z_N(\omega_1 \ldots \omega_N) \) is strictly positive for every \( \omega_1 \ldots \omega_N \) and \( \mathbb{E}Z_N = 1 \). Then we can define a new measure \( \tilde{\mathbb{P}} \) by

\[
\tilde{\mathbb{P}}(\omega_1 \ldots \omega_N) = Z_N(\omega_1 \ldots \omega_N)\mathbb{P}(\omega_1 \ldots \omega_N) \quad \text{for all } \omega_1 \ldots \omega_N. \quad (7.3.1)
\]

We have \( \tilde{\mathbb{P}}(\omega_1 \ldots \omega_N) > 0 \) for every \( \omega_1 \ldots \omega_N \). We can also define \( Y_n = \frac{X_n}{Z_n} \). Under \( \tilde{\mathbb{P}} \), the process \( Y_0, Y_1, \ldots, Y_N \) is a martingale.

**Proof:** Because \( Z_n \) is a martingale, if we set \( Z = Z_N \), then we are in the setting of Theorem 3.2.1 and Lemmas 3.2.5 and 3.2.6 apply. Let \( n \) be a nonnegative integer less than \( N \). Since \( Y_{n+1} \) depends on only the first \( n+1 \) tosses, Lemma 3.2.6 implies

\[
\tilde{\mathbb{E}}_n[Y_{n+1}] = \frac{1}{Z_n} \mathbb{E}_n[Y_{n+1}Z_{n+1}] = \frac{1}{Z_n} \mathbb{E}_n[X_{n+1}].
\]

Since \( X_0, X_1, \ldots, X_N \) is a martingale under \( \mathbb{P} \), we have

\[
\frac{1}{Z_n} \mathbb{E}_n[X_{n+1}] = \frac{1}{Z_n} X_n = Y_n.
\]
Therefore,
\[ \overline{E}_n[Y_{n+1}] = Y_n, \]
which is the martingale property for \( Y_0, Y_1, \ldots, Y_N \) under \( \overline{P} \). □

**Remark 7.3.2** Let \( Z_n, n = 0, 1, \ldots, N, \) be a martingale under a probability measure \( \mathbb{P} \) as described in Theorem 7.3.1. It is typically not the case that \( \frac{1}{Z_n}, n = 0, 1, \ldots, N, \) is also a martingale under \( \mathbb{P} \). In fact, from Jensen’s inequality applied to the convex function \( \varphi(x) = \frac{1}{x} \), defined for \( x > 0 \), and from the fact that the expected value of martingale does not vary with time so that \( Z_0 = \mathbb{E}[Z_N] \), we see that
\[ \frac{1}{Z_0} = \varphi(Z_0) = \varphi(\mathbb{E}[Z_N]) \leq \mathbb{E}[\varphi(Z_N)] = \mathbb{E}\left[ \frac{1}{Z_N} \right]. \] (7.3.2)

Because \( \varphi''(x) > 0 \) for all \( x > 0 \), the function \( \varphi \) has no “flat spots,” so inequality (7.3.2) is strict unless \( Z_N(\omega_1 \ldots \omega_n) = Z_0 = 1 \) for every \( \omega_1 \ldots \omega_N \). But if this were the case, then we would have \( Z_n = \mathbb{E}_n[Z_N] = 1 \) for all \( n = 0, 1, \ldots, N, \) and the martingale \( Z_n \) would just be the constant \( 1 \) for all values of \( n \) and all outcomes of the coin tossing. Except for this case, the inequality in (7.3.2) is strict, and because the expected value of a martingale cannot vary with \( n \), we conclude that \( \frac{1}{Z_n}, n = 0, 1, \ldots, N, \) is not a martingale under \( \mathbb{P} \). However, \( \frac{1}{Z_n}, n = 0, 1, \ldots, N, \) is a martingale under the probability measure \( \overline{P} \) given by (7.3.1). This is just a special case of Theorem 7.3.1 in which the martingale \( X_n \) is \( 1 \) for all values of \( n \) and all outcomes of the coin tossing. □

We now apply Remark 7.3.2 to the foreign exchange model of Section 7.1. In that section, the risky asset denominated in domestic currency was the foreign money market account converted to domestic currency, which we called \( S_n = M_n^f Q_n \). We then discounted at the domestic interest rate to obtain \( \frac{S_n}{M_n} = \frac{M_n^f Q_n}{M_n} \). We constructed the risk-neutral probability measure \( \overline{P} \), so that
\[ \frac{S_n}{M_n} = \frac{M_n^f Q_n}{M_n}, \quad n = 0, 1, \ldots, N, \] (7.3.3)
is a martingale under \( \overline{P} \). This was done using the risk-neutral probabilities \( \overline{p} \) and \( \overline{q} \) defined by (7.1.8).

The risky asset dominated in foreign currency was the domestic money market converted to foreign currency, which we called \( W_n = \frac{M_n}{Q_n} \). We then
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discounted at the foreign interest rate to obtain \( \frac{W_n}{M'_n} = \frac{M_n}{M'_n Q_n} \). We constructed the risk-neutral probability measure \( \tilde{\mathbb{P}}^f \) so that

\[
\frac{W_n}{M'_n} = \frac{M_n}{M'_n Q_n}, \quad n = 0, 1, \ldots, N,
\]

is a martingale under \( \tilde{\mathbb{P}}^f \).

The expression in (7.3.4) is the reciprocal of the expression in (7.3.3). According to Remark 7.3.2, the reciprocal of a martingale is not a martingale unless we change the probability measure. We can apply this idea to change from the measure \( \tilde{\mathbb{P}} \) under which \( \frac{S_n}{M_n} \) is a martingale to the measure \( \tilde{\mathbb{P}}^f \) under which \( \frac{W_n}{M'_n} \) is a martingale. In fact, we use as the \( Z_n \) process the martingale \( \frac{S_n}{M_n} \) itself, except that we must normalize it so that it has expected value 1 at time \( N \), as required by Theorem 7.3.1. We thus define

\[
Z_n = \frac{S_n}{S_0 M_n} = \frac{M'_n Q_n}{M_n Q_0}, \quad n = 0, 1, \ldots, N.
\]

Because \( Z_n, n = 0, 1, \ldots, N \), is a martingale under \( \tilde{\mathbb{P}} \), its expected value under \( \tilde{\mathbb{P}} \) does not depend on \( n \), and hence we have

\[
\tilde{\mathbb{E}}[Z_N] = Z_0 = 1
\]

because \( M_0 = 1 \) and \( M'_0 = 1 \).

Lemmas 3.2.5 and 3.2.6 now take the form of the following lemmas. Note that here we are using the martingale \( Z_n \) to change from \( \tilde{\mathbb{P}} \) to \( \tilde{\mathbb{P}}^f \), whereas in Lemmas 3.2.5 and 3.2.6 we were changing from \( \mathbb{P} \) to \( \tilde{\mathbb{P}} \), but otherwise the statements of the following lemmas are just those of Lemmas 3.2.5 and 3.2.6.

**Lemma 7.3.3** Let \( n \) be a positive integer between 0 and \( N \), and let \( Y \) be a random variable depending only on the first \( n \) coin tosses. Then

\[
\tilde{\mathbb{E}}^f[Y] = \tilde{\mathbb{E}}[Z_n Y],
\]

where \( \tilde{\mathbb{E}}^f \) is the expected value using the probability measure \( \tilde{\mathbb{P}}^f \).

**Lemma 7.3.4** Let \( n \leq m \) be positive integers between 0 and \( N \), and let \( Y \) be a random variable depending only on the first \( m \) coin tosses. Then

\[
\tilde{\mathbb{E}}^f_n[Y] = \frac{1}{Z_n} \tilde{\mathbb{E}}[Z_m Y],
\]

where \( \tilde{\mathbb{E}}^f_n \) is the conditional expected value using the probability measure \( \tilde{\mathbb{P}}^f \).
We could of course make the change of measure in the reverse direction. The process \( \frac{1}{Z_n}, n = 0, 1, \ldots, N \), is a martingale under \( \tilde{P}^f \), and so we can use it as the “\( Z_n \)” process in Remark 7.3.2, and thereby change from the probability measure \( \tilde{P}^f \) to the probability measure \( \tilde{P} \). The resulting formulas are given by the following lemmas.

**Lemma 7.3.5** Let \( n \) be a positive integer between 0 and \( N \), and let \( Y \) be a random variable depending only on the first \( n \) coin tosses. Then

\[
\tilde{E}[Y] = \tilde{E}^f \left[ \frac{Y}{Z_n} \right].
\]  

(7.3.9)

**Lemma 7.3.6** Let \( n \leq m \) be positive integers between 0 and \( N \), and let \( Y \) be a random variable depending only on the first \( m \) coin tosses. Then

\[
\tilde{E}_n[Y] = Z_n \tilde{E}^f_n \left[ \frac{Y}{Z_m} \right],
\]  

(7.3.10)

**Example 7.3.7** We illustrate Lemma 7.3.6 using the numerical example of Section 7.3. We take \( n = 1, m = 2 \) and

\[
Y(\text{HH}) = 8, \ Y(\text{HT}) = 4, \ Y(\text{TH}) = 16, \ Y(\text{TT}) = 8.
\]

We compute \( \tilde{E}_1[Y](H) \). Under \( \tilde{P} \), the probability of \( H \) on the second toss is (see (7.2.6)) \( \tilde{p} = \frac{13}{40} \) and the probability of \( T \) is \( \tilde{q} = \frac{27}{40} \). Therefore,

\[
\tilde{E}_1[Y](H) = \tilde{p} Y(\text{HH}) + \tilde{q} Y(\text{HT}) = \frac{13}{40} \cdot 8 + \frac{27}{40} \cdot 4 = \frac{26}{10} + \frac{27}{10} = \frac{53}{10}.
\]

To compute the right-hand side of (7.3.10), we need some values of \( Z_1 \) and \( Z_2 \). The relevant values are

\[
Z_1(H) = \frac{M_1^f Q_1(H)}{M_1 Q_0} = \frac{\frac{4}{4} \cdot 3 \cdot 2}{8} = \frac{3}{5},
\]

\[
Z_2(\text{HH}) = \frac{M_2^f Q_2(\text{HH})}{M_2 Q_0} = \frac{\frac{16}{16} \cdot \frac{9}{2}}{64} = \frac{64}{25}.
\]

\[
Z_2(\text{HT}) = \frac{M_2^f Q_2(\text{HT})}{M_2 Q_0} = \frac{\frac{16}{16} \cdot \frac{9}{2}}{256} = \frac{256}{225}.
\]
7.4. APPLICATIONS

Under \( \tilde{P}^f \), the probability of \( H \) on the second toss is (see (7.2.8)) \( \tilde{p}^f = \frac{13}{25} \) and the probability of \( T \) is \( \tilde{q}^f = \frac{12}{25} \). Therefore,

\[
Z_1(H) \tilde{E}_2 \left[ \frac{Y}{Z_m} \right] = Z_1(H) \left[ \frac{\tilde{p}^f Y(\text{HH})}{Z_2(\text{HH})} + \tilde{q}^f Y(\text{HT}) \right] \\
= \frac{8}{5} \cdot \left[ \frac{13}{25} \cdot \frac{8}{25} + \frac{12}{25} \cdot \frac{4}{256} \right] \\
= \frac{8}{5} \cdot \left[ \frac{13}{8} + \frac{27}{16} \right] = \frac{8}{5} \cdot \frac{53}{16} = \frac{53}{10}.
\]

The two-sides of (7.3.10) agree.

7.4 Applications

7.4.1 Forward Exchange Rates

Let \( n \) and \( m \) be integers satisfying \( 0 \leq n \leq m \leq N \). The time-\( n \) forward exchange rate for foreign currency per unit of domestic currency for delivery at time \( m \) is a random variable \( \text{For}_{n,m} \) depending on only the first \( n \) coin tosses. This random variable is chosen so that a contract to receive one unit of foreign currency in exchange for a payment of \( \text{For}_{n,m} \) in domestic currency at time \( m \) has value zero at time \( n \). According to the risk-neutral pricing formula, the contract just described has time-\( n \) value

\[
\tilde{E}_n \left[ \frac{1}{(1 + r)^{m-n}} (Q_m - \text{For}_{n,m}) \right]. \tag{7.4.1}
\]

Because \( \text{For}_{n,m} \) depends on only the first \( n \) tosses, \( \tilde{E}_n[\text{For}_{n,m}] = \text{For}_{n,m} \). In order for the expression in (7.4.1) to be zero, we must therefore have

\[
\text{For}_{n,m} = \tilde{E}_n[Q_m].
\]

Because \( \frac{M^f_j Q_n}{M^f_n} \) is a martingale under \( \tilde{P} \) (see (7.1.11)), we have

\[
\text{For}_{n,m} = \tilde{E}_n[Q_m] = \frac{M_m}{M^f_m} \tilde{E}_n \left[ \frac{M^f_j Q_m}{M^f_m} \right] = \frac{M_m}{M^f_m} \cdot \frac{M^f_j Q_n}{M^f_n} = \left( 1 + \frac{r}{1 + rf} \right)^{m-n} Q_n.
\]

We summarize with the following definition.
Definition 7.4.1 Let \( n \) and \( m \) be integers satisfying \( 0 \leq n \leq m \leq N \). The time-\( n \) forward exchange rate for foreign currency per unit of domestic currency for delivery at time \( m \) is

\[
For_{n,m} = \left( \frac{1 + r}{1 + r_f} \right)^{m-n} Q_n.
\]  (7.4.2)

The time-\( n \) forward exchange rate for domestic currency per unit of domestic currency for delivery at time \( m \) is a random variable \( For^f(n, m) \) depending on only the first \( n \) coin tosses. This random variable is chosen so that the contract to receive one unit of domestic currency in exchange for a payment of \( For^f_{n,m} \) in foreign currency at time \( m \) has value zero at time \( n \). According to the risk-neutral pricing formula, the contract just described has time-\( n \) value (priced in foreign currency)

\[
\tilde{E}^f_n \left[ \frac{1}{(1 + r_f)^{m-n}} \left( \frac{1}{Q_m} - For^f_{n,m} \right) \right].
\]  (7.4.3)

It is left as an exercise to show that this contract has value zero at time \( n \) if and only if \( For^f_{n,m} \) is given by the following definition.

Definition 7.4.2 Let \( n \) and \( m \) be integers satisfying \( 0 \leq n \leq m \leq N \). The time-\( n \) forward exchange rate for domestic currency per unit of foreign currency for delivery at time \( m \) is

\[
For^f_{n,m} = \left( \frac{1 + r_f}{1 + r} \right)^{m-n} \frac{1}{Q_n}.
\]  (7.4.4)

Example 7.4.3 Let \( 0 \leq n \leq N \) be given. What is the time-\( n \) value in domestic currency of a contract to receive one unit of foreign currency at time \( N \) in exchange for a payment of \( F_{0,N}^f \)?

To solve this problem, we apply the risk-neutral pricing formula. We also use the fact that \( \left( \frac{1 + r_f}{1 + r} \right)^n Q_n \) is a martingale under \( \tilde{P} \) (see (7.1.11)), which means that the average growth rate of \( Q_n \) under \( \tilde{P} \) is \( \frac{1 + r_f}{1 + r} \) per period.
The time-$n$ value of the contract described is

$$V_n = \tilde{E}_n \left[ \frac{1}{(1+r)^{N-n}} (Q_N - \text{For}_{0,N}) \right]$$

$$= \frac{1}{(1+r)^{N-n}} \tilde{E}_n [Q_N] - \frac{1}{(1+r)^{N-n}} \text{For}_{0,N}$$

$$= \frac{1}{(1+r)^{N-n}} \left( \frac{1+r}{1+r^f} \right)^{N-n} Q_n - \frac{1}{(1+r)^{N-n}} \left( \frac{1+r}{1+r^f} \right)^N Q_0$$

$$= \frac{1}{(1+r^f)^{N-n}} Q_n - \frac{(1+r)^n}{(1+r^f)^N} Q_0.$$ 

If $n = 0$, this is zero, as it should be, since the forward price in this example is the one set at time zero. However, if $n$ is not zero, then this contract typically has a nonzero value, and this value could be either positive or negative.

**Example 7.4.4** Let $0 \leq n < m \leq N$ be given. Consider a contract to receive $\text{For}_{m-1,m}$ paid in foreign currency at time $m$. What is the value in foreign currency of this contract at time $n$?

To solve this problem, we apply the risk-neutral pricing formula under $\tilde{P}_f$. We also use the fact that $(\frac{1+r}{1+r^f})^n \frac{1}{Q_n}$ is a martingale under $\tilde{P}_f$ (see (7.1.18)), which means that the average growth rate of $\frac{1}{Q_n}$ under $\tilde{P}_f$ is $\frac{1+r^f}{1+r}$. The time-zero value of the contract described is

$$V_n^f = \tilde{E}_n^f \left[ \frac{1}{(1+r^f)^{m-n}} \text{For}_{m-1,m} \right]$$

$$= \tilde{E}_n^f \left[ \frac{1}{(1+r^f)^{m-n}} \cdot \frac{1+r^f}{1+r} \cdot \frac{1}{Q_{m-1}} \right]$$

$$= \frac{1}{(1+r^f)^{m-n-1}(1+r)} \tilde{E}_n^f \left[ \frac{1}{Q_{m-1}} \right]$$

$$= \frac{1}{(1+r^f)^{m-n-1}(1+r)} \cdot \left( \frac{1+r^f}{1+r} \right)^{m-n-1} \frac{1}{Q_n}$$

$$= \frac{1}{(1+r)^{m-n} Q_n}.$$ 

**7.4.2 Put-Call Duality**

In this subsection we consider call and put options on currency.
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We begin with a contract that grants the option to buy one unit of foreign currency in exchange for \( K \) units of domestic currency at time \( N \). In other words, this contract is a call paying \( (Q_N - K)^+ \) units of domestic currency at time \( N \). For \( 0 \leq n \leq N \), the time-\( n \) domestic currency value of this contract is

\[
C_n = \tilde{E}_n \left[ \frac{1}{(1 + r)^{N-n}} (Q_N - K)^+ \right] \quad (7.4.5)
\]

We also consider a contract that grants the option to sell one unit of domestic currency in exchange of \( \frac{1}{K} \) units of foreign currency at time \( N \). In other words, this second contract is a put paying \( (\frac{1}{K} - \frac{1}{Q_N})^+ \) units of foreign currency at time \( N \). For \( 0 \leq n \leq N \), the time-\( n \) foreign currency value of this contract is

\[
P_n = \tilde{E}_f \left[ \frac{1}{(1 + r_f)^{N-n}} \left( \frac{1}{K} - \frac{1}{Q_N} \right)^+ \right]. \quad (7.4.6)
\]

**Theorem 7.4.5** We have

\[ C_n = KQ_n P_n. \quad (7.4.7) \]

**Proof:** We recall the process \( Z_n = \frac{M_n Q_n}{M_n Q_0} \) of (7.3.5) that permits us to change from the probability measure \( \tilde{P} \) to the probability measure \( \tilde{P}f \). Using this process and Lemma 3.2.6, we may write

\[
KQ_n P_n = KQ_n \tilde{E}_n \left[ \frac{1}{(1 + r_f)^{N-n}} \left( \frac{1}{K} - \frac{1}{Q_N} \right)^+ \right]
\]

\[
= KQ_n \frac{1}{Z_n} Z_N \frac{M_n^f}{M_N^f} \left( \frac{1}{K} - \frac{1}{Q_N} \right)^+ \left[ \frac{M_n^f Q_N}{M_N Q_0} \cdot \frac{M_n^f}{M_N} \left( \frac{1}{K} - \frac{1}{Q_N} \right)^+ \right]
\]

\[
= M_n \tilde{E}_n \left[ \frac{KQ_N}{M_N} \left( \frac{1}{K} - \frac{1}{Q_N} \right)^+ \right]. \quad (7.4.8)
\]

There are two cases to consider. If \( \frac{1}{K} - \frac{1}{Q_N} \geq 0 \), then

\[
KQ_N \left( \frac{1}{K} - \frac{1}{Q_N} \right)^+ = KQ_N \left( \frac{1}{K} - \frac{1}{Q_N} \right) = Q_N - K,
\]
and this expression is also nonnegative, so $Q_N - K = (Q_N - K)^+$. In other words,

$$KQ_N\left(\frac{1}{K} - \frac{1}{Q_N}\right)^+ = (Q_N - K)^+.$$  \hspace{1cm} (7.4.9)

On the other hand, if $\frac{1}{K} - \frac{1}{Q_N} < 0$, then $Q_N - K < 0$ and we again have (7.4.9), this time with both sides equal to zero. Substituting (7.4.9) into (7.4.8), we obtain

$$KQ_nP_n = M_n\tilde{E}_n\left[\frac{1}{M_N}(Q_N - K)^+\right] = \tilde{E}_n\left[\frac{1}{(1+r)^{N-n}}(Q_N - K)^+\right] = C_n.$$  \hspace{1cm} □

We can understand Theorem 7.4.5 in the following way. The call is the right to exchange $K$ units of domestic currency for one unit of foreign currency, and this would be exercised when one unit of foreign currency is more valuable than $K$ units of domestic currency. We could think of the call as $K$ contracts, each granting the right to exchange 1 unit of domestic currency for $\frac{1}{K}$ units of foreign currency, and this would be exercised when $\frac{1}{K}$ units of foreign currency is more valuable than one unit of domestic currency. This is therefore the same as $K$ contracts granting the right to sell one unit of domestic currency for $\frac{1}{K}$ units of foreign currency, which is $K$ puts on the domestic currency. Thus, one call on foreign currency with strike price $K$ is worth $K$ puts on domestic currency with strike price $\frac{1}{K}$. However, the call is naturally denominated in domestic currency and the $K$ puts are naturally denominated in foreign currency. To write an equation relating these two types of contracts, we need to also convert the currency at the time of pricing. The term $Q_n$ in (7.4.7) is present to make this conversion.

### 7.5 Replicating portfolio for a forward exchange contract

In this section we return to the forward exchange rates of Subsection 7.4.1 and discuss how to construct a portfolio that replicates a forward exchange contract. To begin this discussion, we must first find an equation that describes the evolution of the value of a portfolio that trades in the domestic money market account and the foreign money market account. We begin with an example.
Example 7.5.1 Consider the exchange rate process in Figure 7.2.1. As in Section 7.4.1, we take the domestic and foreign interest rates to be $r = \frac{1}{4}$ and $r^f = \frac{1}{3}$, respectively. To make matters more specific, we assume that the domestic currency is dollars (denoted $\$ ) and the foreign currency is British pounds (denoted £). The time-zero exchange rate is $\$Q_0/£ = \$2/£$, which we read as “two dollars per pound.” We begin with initial capital $\$X_0 = \$0$ and purchase £1 at time zero, which we invest in the British money market. To do this, we must borrow $\$2$ from the U.S. money market. When we go to time one, the investment in the British money market grows to

$$(1 + r^f) \cdot £1 = \frac{4}{3} \cdot £1 = £\frac{4}{3}.$$ 

The debt in the U.S. money market, which we call $-\$2$ because it is a debt, grows to

$$(1 + r) \cdot $(-2) = \frac{5}{4} \cdot $(-2) = $\left( -\frac{5}{2} \right).$$

To determine the value of the portfolio at time one, we must convert to a common currency. Since we are recording the portfolio value in dollars, we convert the pounds to dollars. But the exchange rate depends on the first coin toss, and we thus get two possible portfolio values at time one, which are

$$X_1(H) = £\frac{4}{3} \cdot \$3/£ + $\left( -\frac{5}{2} \right) = $4 + $\left( -\frac{5}{2} \right) = $\frac{3}{2},$$

$$X_1(T) = £\frac{4}{3} \cdot \$\frac{4}{3}/£ + $\left( -\frac{5}{2} \right) = $\frac{16}{9} + $\left( -\frac{5}{2} \right) = $\left( -\frac{13}{18} \right).$$

We see that even though the British interest rate is higher than the U.S. interest rate, borrowing dollars in the U.S. in order to invest in pounds in the UK can lead to a loss.

In general, the computation of $X_1$ proceeds as follows. If we begin with initial capital $\$X_0$ and purchase $\Delta_0$ pounds, then we will need to pay £$\Delta_0 \cdot $Q_0/£ = £$\Delta_0$Q_0, and this is subtracted from our initial capital, leaving us with $(X_0 - \Delta_0 Q_0)$. This might be negative, in which case it represents borrowing in the U.S. money market and paying the U.S. interest rate $r$. It could also be positive, in which case it represents investing in the U.S. money market and receiving the U.S. interest rate $r$. In either case, when we go to time one, the cash in dollars changes from $(X_0 - \Delta_0 Q_0)$ to

$$(1 + r)(X_0 - \Delta_0 Q_0).$$

(7.5.1)
This is the change from $(-2)$ to $(-5)$ in the numerical case above where $X_0 = 0$, $\Delta_0 = 1$, $Q_0 = 2$, and $1 + r^f = \frac{5}{3}$. On the other hand, the $\Delta_0$ pounds purchased at time zero is invested at the foreign interest rate and grows to be $(1 + r^f)\Delta_0$ pounds at time one. This is the £$\frac{4}{3}$ in the example above, where $\Delta_0 = 1$ and $1 + r^f = \frac{4}{3}$. We convert this to dollars by multiplying by $\$Q_1/£$, which results in $(1 + r^f)\Delta_0 Q_1$. (7.5.2)

The sum of (7.5.1) and (7.5.2) is the value in dollars of the portfolio at time one. In other words, the value in dollars of the portfolio at time one is

$$X_1 = (1 + r^f)\Delta_0 Q_1 + (1 + r)(X_0 - \Delta_0 Q_0).$$

(7.5.3)

Because $Q_1$ depends on the first coin toss, the value of the portfolio at time one also depends on the first coin toss (unless $\Delta_0 = 0$, in which case $Q_1$ drops out of (7.5.3)).

An equation analogous to (7.5.3) holds when we move from time $n$ to time $n+1$. Suppose at time $n$ we decide to hold $\Delta_n$ pounds, selling or buying pounds as necessary to achieve this when we arrive at time $n$. Any money left over is invested at the U.S. interest rate $r$, and if there is insufficient money to pay for $\Delta_n$ pounds, we borrow at the U.S. interest rate $r$. Then the value $X_{n+1}$ of the portfolio at time $n+1$, denominated in dollars, and the value $X_n$ of the portfolio at time $n$, also denominated in dollars, are related by the equation

$$X_{n+1} = (1 + r^f)\Delta_n Q_{n+1} + (1 + r)(X_n - \Delta_n Q_n)$$

$$= \Delta_n((1 + r^f)Q_{n+1} - (1 + r)Q_n) + (1 + r)X_n.$$  (7.5.4)

In (7.5.4), $\Delta_n$ is permitted to depend on the first $n$ coin tosses. Then $X_n$ will depend on only the first $n$ coin tosses and $X_{n+1}$ will depend on only the first $n + 1$ coin tosses. In other words, both $\Delta_0, \Delta_1, \ldots, \Delta_{N-1}$ and $X_0, X_1, \ldots, X_N$ are adapted processes.

**Theorem 7.5.2** The process $\frac{X_n}{(1+r)^n}$, $n = 0, 1, \ldots, N$, is a martingale under the domestic risk-neutral measure $\tilde{P}$.

**Proof:** Recall from (7.1.11) that $\left(\frac{1+r^f}{1+r}\right)^n Q_n$, $n = 0, 1, \ldots, N$, is a martingale under $\tilde{P}$. Therefore, for $0 \leq n \leq N - 1$,

$$\tilde{E}_n \left[ \left(\frac{1+r^f}{1+r}\right)^{n+1} Q_{n+1} \right] = \left(\frac{1+r^f}{1+r}\right)^n Q_n.$$
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We can factor the nonrandom quantity \((1 + r f)^n\) out of the conditional expectation on the left-hand side to obtain

\[
(1 + r f)^n \mathbb{E}_n \left[ \frac{1 + r f}{(1 + r)^{n+1}} Q_{n+1} \right] = \left( \frac{1 + r f}{1 + r} \right)^n Q_n,
\]

which simplifies to

\[
\mathbb{E}_n \left[ \frac{1 + r f}{(1 + r)^{n+1}} Q_{n+1} \right] = \frac{Q_n}{(1 + r)^n}.
\] (7.5.5)

From the first line of (7.5.4) we have

\[
\frac{X_{n+1}}{(1 + r)^{n+1}} = \frac{1 + r f}{(1 + r)^{n+1}} \Delta_n Q_n + \frac{X_n}{(1 + r)^n} - \Delta_n \frac{Q_n}{(1 + r)^n}.
\]

Using linearity of conditional expectations, taking out what is known, and using (7.5.5), we obtain

\[
\mathbb{E}_n \left[ \frac{X_{n+1}}{(1 + r)^{n+1}} \right] = \Delta_n \mathbb{E}_n \left[ \frac{1 + r f}{(1 + r)^{n+1}} Q_n \right] + \mathbb{E}_n \left[ \frac{X_n}{(1 + r)^n} \right] - \mathbb{E}_n \left[ \Delta_n \frac{Q_n}{(1 + r)^n} \right]
\]

\[
= \Delta_n \frac{Q_n}{(1 + r)^n} + \frac{X_n}{(1 + r)^n} - \Delta_n \frac{Q_n}{(1 + r)^n}
\]

\[
= \frac{X_n}{(1 + r)^n} - \Delta_n \frac{Q_n}{(1 + r)^n}.
\]

This is the martingale property for \(\frac{X_n}{(1 + r)^n}\). \(\Box\)

Consider a forward contract to purchase one unit of foreign currency for \(\text{For}_{0,N} = \left( \frac{1 + r f}{1 + r} \right)^N Q_0\) units of domestic currency at time \(N\). The formula for \(\text{For}_{0,N}\) is (7.4.2), and the formula is chosen so that at time zero this contract has value zero. If we take a short position in this contract (i.e., we agree to deliver a unit of foreign currency at time \(N\) in exchange for the payment \(\left( \frac{1 + r f}{1 + r} \right)^N Q_0\) in domestic currency), we receive

\[
X_0 = 0
\] (7.5.6)
at time zero and we are obligated to pay $Q_N - \left( \frac{1+r}{1+r_f} \right)^N Q_0$ at time $N$. (This payment may be negative, in which case we actually receive money at time $N$. If there were no possibility that we would receive money at time $N$, we would not accept the short position in this contract without demanding an initial payment.) Starting at $X_0 = 0$, we want to determine random variables $\Delta_0, \Delta_1, \ldots, \Delta_{N-1}$, each $\Delta_n$ depending on only the first $n$ coin tosses, so that regardless of how the tossing turns out, the final value of our portfolio is

$$X_N = Q_N - \left( \frac{1+r}{1+r_f} \right)^N Q_0.$$  

(7.5.7)

In other words, we want to replicate the payoff of the forward contract.

The method we use to determine $\Delta_n$ is to first use the martingale property of Theorem 7.5.2 to obtain a formula for $X_n$. Once this formula is obtained, we can replace $n$ by $n+1$ to obtain a formula for $X_{n+1}$. We then substitute both these formulas into (7.5.4) and solve for $\Delta_n$.

We now use the martingale property of Theorem 7.5.2. We proved “the one-step ahead” version of the martingale property in that theorem, but this implies the “multiple-step ahead” version. In particular, the martingale property proved in Theorem 7.5.2 implies that for all $n = 0, 1, \ldots, N$,

$$\frac{X_n}{(1+r)^n} = \mathbb{E}_n \left[ \frac{X_N}{(1+r)^N} \right].$$  

(7.5.8)

But $X_N$ is given by (7.5.7) and $\left( \frac{1+r_f}{1+r} \right)^N Q_n$ is a martingale under $\tilde{P}$, so

$$\frac{X_n}{(1+r)^n} = \mathbb{E}_n \left[ \frac{Q_N}{(1+r)^N} - \frac{Q_0}{(1+r_f)^N} \right]$$

$$= \mathbb{E}_n \left[ \frac{Q_N}{(1+r)^N} \right] - \frac{Q_0}{(1+r_f)^N}$$

$$= \frac{1}{(1+r_f)^N} \mathbb{E}_n \left[ \left( \frac{1+r_f}{1+r} \right)^N Q_N \right] - \frac{Q_0}{(1+r_f)^N}$$

$$= \frac{1}{(1+r_f)^N} \left( \frac{1+r_f}{1+r} \right)^n Q_n - \frac{Q_0}{(1+r_f)^N}.$$  

Solving this equation for $X_n$ yields

$$X_n = \frac{Q_n}{(1+r_f)^N-n} - \frac{(1+r)^n Q_0}{(1+r_f)^N}, \quad n = 0, 1, \ldots, N.$$  

(7.5.9)
Note that if we substitute \( n = 0 \) into (7.5.9), we obtain (7.5.6), and if we substitute \( n = N \), we obtain (7.5.7).

Equation (7.5.9) with \( n + 1 \) replacing \( n \) is

\[
X_{n+1} = \frac{Q_{n+1}}{(1 + r^f)^{N-n-1}} - \frac{(1 + r)^{n+1}Q_0}{(1 + r^f)^N}, \quad n = 0, 1, \ldots, N - 1. \quad (7.5.10)
\]

We substitute this formula for \( X_{n+1} \) into the left-hand side of (7.5.4) and substitute formula (7.5.9) for \( X_n \) into the second line of the right-hand side of (7.5.4). This results in the equation

\[
\Delta_n \left( (1 + r^f)Q_{n+1} - (1 + r)Q_n \right) + \frac{(1 + r^f)Q_0}{(1 + r^f)^{N-n}} - \frac{(1 + r)^{n+1}Q_0}{(1 + r^f)^N}.
\]

We cancel the term \(-\frac{(1+r)^{n+1}Q_0}{(1+r^f)^N}\), which appears on both sides, and then move all the terms involving \( Q_{n+1} \) to the left-hand side of the equation. This results in

\[
(1 + r^f)Q_{n+1} \left( \frac{1}{(1 + r^f)^{N-n}} - \Delta_n \right) = (1 + r)Q_n \left( \frac{1}{(1 + r)^{N-n}} - \Delta_n \right). \quad (7.5.11)
\]

We now argue from this equation that

\[
\frac{1}{(1 + r^f)^{N-n}} - \Delta_n = 0. \quad (7.5.12)
\]

If this were not the case, then the left-hand side of (7.5.11) would depend on the first \( n + 1 \) coin tosses while the right-hand side would depend on only the first \( n \) tosses. In such a case, the two sides cannot be equal. The only way for the two sides to be equal is for (7.5.12) to hold, in which case both sides are equal to zero. We conclude that (7.5.12) holds, or equivalently,

\[
\Delta_n = \frac{1}{(1 + r^f)^{N-n}}, \quad n = 0, 1, \ldots, N - 1. \quad (7.5.13)
\]

It turns out that (7.5.13) mandates a “buy and hold” strategy. At time zero, we should buy

\[
\Delta_0 = \frac{1}{(1 + r^f)^N}.
\]
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units of the foreign currency. This costs us

\[
\frac{Q_0}{(1 + r_f)^N}
\]

units of the domestic currency, an amount that we borrow at the domestic interest rate. At time one, our \( \Delta_0 \) units of the foreign currency have grown at the foreign interest rate to

\[
(1 + r_f)\Delta_0 = \frac{1}{(1 + r_f)^N - 1} = \Delta_1
\]

units of foreign currency. Therefore, we arrive at time one already holding the right amount of foreign currency in order to proceed to time two; we do not need to make any adjustment. At time two, the \( \Delta_1 \) units of foreign currency we had at time one have grown to

\[
(1 + r_f)\Delta_1 = \frac{1}{(1 + r_f)^{N-2}} = \Delta_2
\]

units of foreign currency. Again we do not need to make any adjustment. We continue on like this, allowing our position in the foreign currency to grow at the foreign interest rate and making no adjustments in the position, until at time \( N \), we have

\[
\Delta_N = 1
\]

unit of foreign currency, which has value \( Q_N \) in domestic currency. On the other hand, our initial debt of \( \frac{Q_0}{(1 + r_f)^N} \) in domestic currency has grown from time zero to time \( N \) at the domestic interest rate to become a debt of

\[
\left( \frac{1 + r}{1 + r_f} \right)^N Q_0.
\]

We thus have a portfolio valued at

\[
X_N = Q_N - \left( \frac{1 + r}{1 + r_f} \right)^N Q_0, \quad (7.5.14)
\]

and this is the value of \( X_N \) we desired (see (7.5.7)). Whereas (7.5.7) was simply a desire, a statement of what we wanted our trading strategy to lead to, (7.5.14) is the fulfillment of that desire. We have confirmed that the trading strategy (7.5.13), starting with initial capital \( X_0 = 0 \), leads to the final portfolio value (7.5.14).