PROJECTIVE PLANE

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DEFINITION:
A projective plane is a set of points and lines that satisfies the following three axioms:
A1: Each pair of distinct points determines a line.
A2: Each pair of distinct lines intersects in a point.
A3: There exists a set of four points such that no line contains more than two of them.

CAUTIONS:
First, be careful with "point" and "line" - they are undefined terms, and we have only the axioms to tell us about their properties. In particular, there's nothing about those axioms that says that the line has to be a straight line (remember, curves can be "lines" too in a general sense), and we'll see that when we look at a model.

Second, when we write sets of points that correspond to a geometric model, we may order them in the order they appear on the line for convenience, but in terms of the point set, this is not significant. In the second image below (the "NO" model), we could write the "line" \{1,4,2\} as \{1,2,4\} if we want.

INTERPRETING THE AXIOMS:
A1 tells us that if we have two lines that have two points in common, then they are in fact the same line (and should not both appear in listing of the lines in the model). A line can consist of more than two points, for example, the line \{1, 2, 3\} but the axiom implies that this line is identical to the line \{1,2\}, or the line \{2,3\}, or the line \{1,3\}. If I were creating a listing of distinct lines in the system, I'd only have one of the above in my list.

A1 also implies that if there are two lines, they can either (1) not intersect, or (2) can intersect only in one point. In other words, it rules out the possibility of a geometric model like this:

\[\text{NO}\]

which would have lines \{1,2,3\} and \{1,4,2\} as distinct - not allowed by the axiom because they share points 1 and 2. A1 on the other hand does allow for something like this:

\[\text{YES}\]

Lines \{1,2,3\} and \{4,2,5\} intersect in only one point, and do not violate the axiom.

A2 tells us that any two lines must intersect in a point. This further refines A1 (which allowed for a one point intersection, or no intersection) - we're now adding the condition that "no intersection" is not permissible. There are no parallel lines in this system.

A1 and A2 are the critical axioms for defining the system. A3 is in there to rule out some possibilities that we'd like to avoid. Without A3, we could have a model where the entire system was this:

\[\text{NO}\]

The issue isn't that there's a line in the system with 5 points, the issue is that we're saying this can't be the entire system - there's got to be some more lines in there somewhere. As a system, this does satisfy A1 - any pair of points determines the line \{1,2,3,4,5\} (i.e. the line determined by \{1,2\} = the line determined by \{1,2,3,4,5\} and so on) - they just happen to all determine the same line. It also satisfies A2 in the vacuously true sense: if there were two lines, they'd have to intersect in a point... but there aren't two lines in the system to worry about.

A system like this is a degenerate case - it technically obeys axioms A1 and A2, but a single line system is not very interesting mathematically. Adding in A3 excludes this possibility and forces any system we consider to have some of the points spread out in a plane. There is no way in the above model to select out a set of four points with no line containing more than two of them - for any four points you select, the only line we have would contain all four.

This axiom is also used to rule out the possibility of something like this:

\[\text{NO}\]

where there is a line with \(n\) number of points along it, and only one point off the line. Again, this would satisfy A1 and A2, but it's not very interesting.

THE SMALLEST PROJECTIVE PLANE:
None of the axioms state how many points/lines have to be in the system. One requirement that does follow from the
axioms is that the number of lines must be the same as the number of points. Since two points determine a line, and two
lines determine a point, they are in a sense swappable, and there must be the the same number of each.
The smallest projective plane is the 7-Point Projective Plane (Fano Plane). Here’s a model:

![Diagram of the 7-Point Projective Plane (Fano Plane)](image)

To visually represent the relationship between the sets, one of the “lines”, \(\{2,4,7\}\), appears as a circle.

**EXERCISE:**
Verify for yourself that the 7-point projective plane satisfies the three axioms - consider all possible intersections of pairs
of lines and observe that the intersection is one and only one point (A1 and A2), and find a set of four points that has the
property that no line contains more than two of them.

![Diagram of the line intersections](image)

**OTHER PROJECTIVE PLANES:**
The number of points in a projective plane will always fit the formula
\[n^2 + n + 1\]
where \(n\) is a positive integer called the order of the plane. The 7 point plane has order 2, since evaluating the formula at \(n = 2\) gives a value of 7.
The formula does not imply that every positive integer \(n\) generates a projective plane, but that if we have a projective
plane (with a certain number of points), then there exists an \(n\) that will produce that number. More details on this can be
found in the reference. There are projective planes with \(n = 3\) (13 point) and \(n = 4\) (21 point).

**COOL LINK WITH PRETTY PICTURES:**
http://www.maa.org/editorial/mathgames/mathgames_05_30_06.html
Includes a model for the 21 point plane.

**AND THE OBLIGATORY WIKI REFERENCE:**
http://en.wikipedia.org/wiki/Projective_plane