An overview of Spherical Geometry, as it relates to this course…

Spherical geometry is something we could spend a couple weeks on, and we're not. We're going to spend like, maybe, 20 minutes on it (OK, understatement, but still...we're not going in-depth). Hyperbolic geometry is the one I'd like to dig a little deeper into, but spherical deserves a mention because you need it to complete a certain picture: it represents the "third option" when you're contemplating parallel postulates.

Recall in the introduction that Euclid had five key postulates, and that there was something a bit dodgy about the fifth, called the parallel postulate. The quick, easy, and typical way to categorize the variant geometries is to view them as what happens when you tweak the fifth postulate, and "Euclidean/hyperbolic/spherical" covers all the options:

> **Fifth Postulate (Euclidean):** Given a line l and a point P not on the line, ***exactly one*** line can be drawn through P which is parallel to l.
>
> **Fifth Postulate (hyperbolic):** Given a line l and a point P not on the line, ***two or more*** lines can be drawn through P which is parallel to l.
>
> **Fifth Postulate (spherical):** Given a line l and a point P not on the line, ***no*** lines can be drawn through P which is parallel to l.

That covers any situation, right?

What's a little misleading about this (and so, the topic of this chunk of notes) is that it makes it look like all three of the geometries share a common core, and then go their separate ways. That is and isn't true. They do all share the common set of Euclid's first four postulates, as conceived by Euclid. However

Modern "Euclidean" geometry is not the same as Euclid's "Euclidean" geometry."

And the difference matters here.

Absolute (neutral) geometry and Hilbert

What we've been studying in depth up to this point in Kay is something called absolute geometry, aka neutral geometry or Hilbert's geometry. What Hilbert did was recognize that, while Euclid did a fantastic job of systematizing things for a guy working in the 4th century BC, there were some gaping holes concerning precision of statements, and unstated assumptions lying all over the place. So he went about the task of cleaning things up and formalizing them even more.

In particular, where Euclid has a fairly loose [paraphrased] "given any two points, you can draw a line through them" as a postulate, Hilbert tightened this up, reasonably asking the question "yeah, well how MANY lines can you draw?" That gave us the incidence axioms, and in particular "two points *determine* a line." That is a subtle but significant difference – it's a uniqueness condition. You pick two points, and not only can I draw a line through them, but there's only ONE line I can draw through them – I can't have multiple things called lines that are all distinct from each other, and yet manage to pass through the same two points.

That's something that Euclid's phrasing leaves open, and here's the thing: ***the way "lines" get defined in spherical geometry, they violate the formal incidence axiom***. To repeat, they don't violate Euclid's version (we can pick two points, and draw a line in spherical geometry – no problem there), they violate Hilbert's version (turns out, we can draw more than one line in some situations – two points don't *determine* a line in spherical geometry).

Because spherical geometry violates that first, fundamental incidence axiom, everything else can go screwy as a result. There are more differences than just an alternate parallel postulate. And so, we say

- (Modern) Euclidean and hyperbolic geometries are built on absolute geometry – you start with a core of absolute geometry, with all the axioms/postulates we've incorporated so far, and at some point, you introduce a new postulate on top of the existing structure: a parallel postulate. At that point, Euclidean and hyperbolic split and go down separate tracks, depending on which parallel postulate you adopt. However, all the results that were established for absolute geometry up to that point hold true for both geometries.
- Spherical geometry does NOT share a common core of absolute geometry – it got "broken" at the very first incidence axiom. Because of that, ***all the results of absolute geometry we've established up to this point cannot be assumed to hold true for spherical geometry***. You're asked to play around with something as a quick experiment that will show you immediately that there's a key theorem of absolute geometry that doesn't hold in spherical.

A reasonable question then is "Well, are there any parts of absolute geometry that do still hold in spherical? What does/doesn't work the same?" But, we're not going to go there. That's the point here – to do spherical geometry thoroughly, we would have to completely start over from the ground up and build it, redoing the incidence axioms and re-establishing everything from that spot on.

What *do* we need to know about spherical geometry, then?

Not a whole lot – basically, the things we need to sort out are

1) What *is* a line, then?
2) What's an angle? How do we measure it?
3) What's a triangle?

And we'll leave it at that.

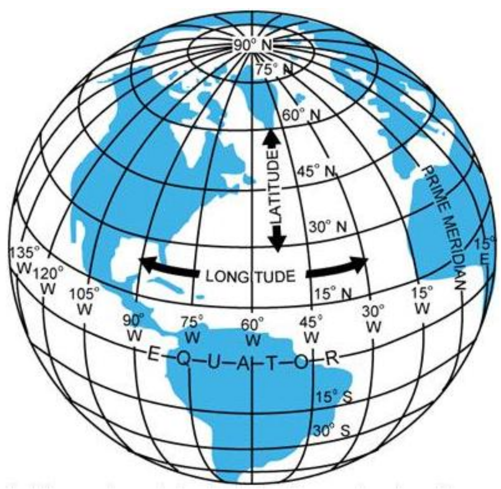
Lines in spherical geometry

For a geometry that is totally weird when you look at it from a formal axiomatic perspective (betweenness goes totally off the rails, for one thing – don't ask!), it's extremely natural from a real world perspective. Literally – it's the geometry that naturally fits navigating the *world*, which is a sphere. Folks have been intuitively using spherical geometry for centuries, and so a look at that intuition can help inform the definition of what a line is and isn't in this context.

If you're doodling on a globe, intuition quickly gives you an idea of something that represents something "line-like" – since one thing we expect lines to do is have no beginning and no end (i.e. they might *pass* through two points, but they keep going indefinitely – they don't start or stop specifically at any two points), we can go "aha! A CIRCLE!" On a globe, a circle has no beginning and no end, since it loops back around to connect with itself, and yet we can also easily imagine starting with any two points and doodling a circle that passes through them.

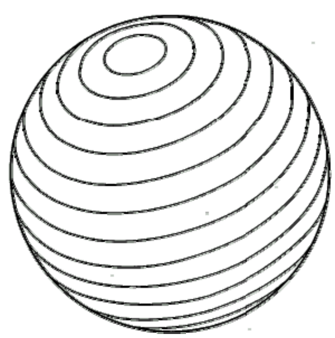
We can also quickly observe that there's more than one way to draw circles on a globe, and now we have to start paying attention to that parallel postulate – we might not want to let every circle that we *could* draw actually count as a "line" in our system.

The two familiar "lines" (which are really circles) are latitude and longitude lines.



Let's see how those work out…

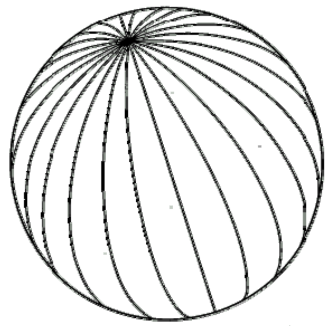
Here are some latitude lines:



Oh, hey, PROBLEM! It is clear that we can draw latitude lines which do not intersect with each other, and are hence parallel. ***We are trying to build a system where there are NO parallel lines.*** Latitude "lines"…you're OUT! You don't count as lines in this system we're trying to build.

So not every type of circle we can draw on a globe gets to be a "line."

How about longitude lines:



Ah, now this shows some promise! All longitude lines eventually intersect with each other at the poles. If we built a geometry solely out of longitude lines, we'd have a geometry where parallel lines could not exist.

The question now is – do we limit ourselves to longitude lines, or can we go broader? How broad can we go in our definition of "line" that will still allow us to hang on to the "there are NO parallel lines, because any two lines will eventually intersect with each other somewhere" criterion?

We can***: longitude lines are a SUBSET of the class of objects that we're going to call "lines" in this system***.

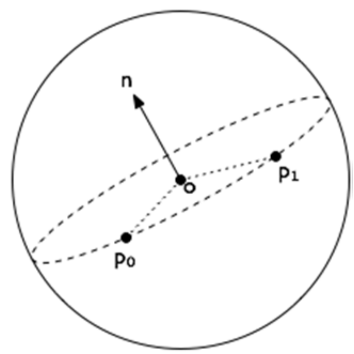
Slice-and-dice: Great Circles

The trick to generalizing the idea behind longitude lines is to think about how you could construct them in the first place. And the trick to *that* is to get creative with the context…

- We're going to define the set of POINTS in our spherical geometry to be "the set of all points on the surface of a sphere" i.e. "the set of all points in 3D space which are a fixed distance (called a radius r) from a given point C called the center."
- Note that there's an oddity right away – to define points in spherical geometry, we're making reference to Euclidean 3D space! This is fine – we can imagine our sphere as floating in the normal 3D space that we're used to. Note also that the center C of our sphere, whose job is to define the sphere, is not itself part of the set of points on the surface of the sphere – so while it defines the geometry, it's not part of the geometry!
- And so, to construct a longitude line in this context
  - Take an orange (a sphere).
  - Arbitrarily pick two points that lie on opposite ends of the *diameter* of the sphere. Call them poles. Mark them.
  - Take a large knife (a plane).
  - Slice right down the center from pole to pole. Notice that you can *angle* your knife through any degree of rotation you like, so there's an infinite number of ways to do this.

- But no matter how you do this, you get a cross section which is a circle, and not only that, it's a circle whose center is the same center as the center of the sphere, and whose diameter is the same diameter as that of the sphere. It's a biggest possible circle (which is where it differs from latitude lines – if you slice laterally without passing end to end through the poles, the diameter of your cross section is related to a *chord* on the sphere that is not its diameter).
- That circle is a longitude line on the globe, right?

- Now, GENERALIZE
  - Take ANY two points on the surface of the sphere (not necessarily the two things you designated to be the poles, and not even necessarily at opposite ends of a diameter).
  - Use the center of the sphere as a third point.
  - Since our sphere is hanging out in normal Euclidean 3D space, those three points are living there too, right? As long as those points aren't living at opposite ends of a diameter, they are a set of three *non-colinear* points. So they determine a plane.



  - That plane is your knife. Slice your orange. The cross section you get will pass through the center of the sphere, and will create a circle whose diameter is the same as the diameter of the sphere. It will also contain the two points on its circumference – they just won't be sitting on opposite sides of the diameter. That thing is a **great circle**.
- Longitude lines are also examples of great circles. The general definition of "great circle" is "any circle on the surface of a sphere that lies in a plane passing through the sphere's center."

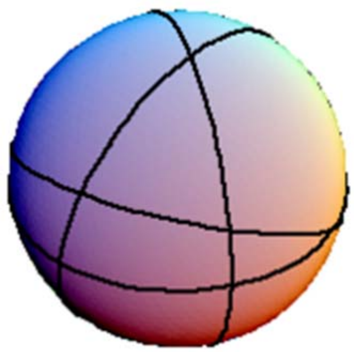
Lines in spherical geometry are defined to be great circles, where great circles are defined by the above constructions. [Yes, there is a more rigorous, formal way to define the above. No, we're not going to go through it all – this is an adequate working definition that captures the essentials.]

Notice that there is a bit of a difference between the situations of "points do lie on opposite ends of a diameter" vs "points don't lie on opposite ends of a diameter." In both cases, you can get a great circle through the points – a circle that that "lies in a plane passing through the sphere's center." The difference is in how many you can get.

- If the two points chosen are *not* on opposite ends of the diameter, that really is a unique line (great circle) going through those points. That's because it's determined by three non-colinear points in Euclidean space, and we know that those three points determine a unique plane.

Unique plane = only one way to slice it to get the cross section = only one great circle that contains those two points.

- If the two points chosen are on opposite ends of the diameter, then those points and the center are now all in a line with each other! We can still make the slice, no problem, but it's not a unique slice. There's an infinite number of great circles containing those two points.
- No matter how you slice it (ha, ha), it is impossible to have parallel lines. ANY two great circles you draw will intersect each other somewhere. In fact, they'll intersect in *more than one point* (two, to be precise – for any intersection you see on the side facing you, there's another one around the back of the thing).



That bit with the great circles does all the things we said up front it would do:

- It satisfies Euclid's Postulate: "pick any two points and you can draw a line passing through them." CHECK.
- It violates Hilbert's Axiom: "pick any two points and you can draw a UNIQUE line passing through them." FAIL. Certain pairs of points have multiple lines which can be drawn.
- It satisfies the parallel postulate variation we're aiming for: "Given a line l and a point P not on the line, ***no*** lines can be drawn through P which is parallel to l." That's correct. Draw a great circle. Pick another point not on it. Try drawing another great circle through that point that *doesn't* intersect the one you've already got. Good luck. Not gonna happen.

Building on from there…

"Line" was the tricky part. "Angle" and "triangle" follow pretty easily (although there's some wackiness with "angle" that we're going to wave our hands over).

So, we've got "line." How about "line segment"? Easily enough – just clip your line off at the two endpoints. The segment is the set of points between the ends (and just use an intuitive idea of "between" here, because the formal version turns out to be a headache).

What about "ray"? Well, now, that's a bit of a problem. In Euclidean plane geometry, you get a ray by clipping a line at one end, and keeping going in the other direction. In spherical, if you clip at one end, but then keep going in the other direction…you eventually wrap back around, come full circle, and pass right back through the point you were calling the "endpoint." You just drew a line!

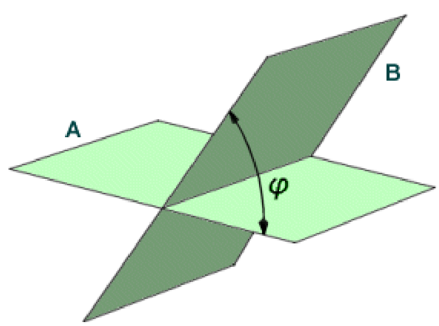
There is still a way to define “ray,” sort of, but it’s not intuitive – you have to start jumping back and forth between the great circles and the planes that were used to make the slices that created the great circles. These “rays” live outside the points on the sphere, don’t coincide with the lines, and really aren’t part of the geometry.

That in turn screws up “angle”. This is another spot where the intuitive idea is intuitive, but the formal version is clunky.

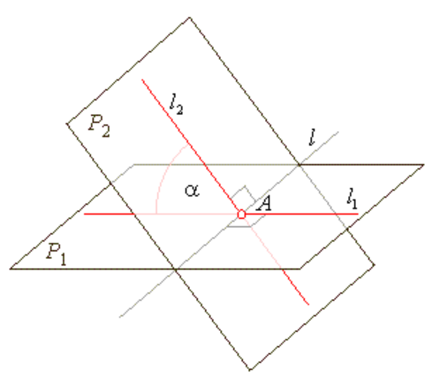
So, when we talk about angles in a mathematical context, are we talking about the point set formed by the union of rays, or are we talking about the gap between them?

In 2D plane geometry, this is clear cut. An angle, as a geometric object, is the POINT SET formed by the union of two rays. We define a MEASURE on this particular object that effectively measures the gap between them…but we don’t say the gap *is* the angle.

However, there’s plenty of familiar mathematical contexts where we’re sloppy with that. In Multivariable Calculus, I’ve got a section on “finding the angle between two planes.” Think about that one for a second – planes aren’t rays. The point sets which are the planes are not the point sets that form an angle. When we talk about measuring the angle between the two planes, we’re talking about measuring the gap between them, and the key here is *everybody gets what you’re talking about without ever going into a convoluted dance about just which rays in the planes form the geometric object called “angle”.*

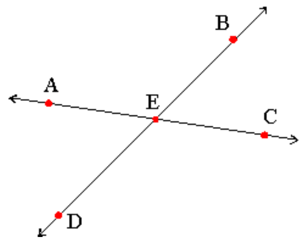


The rays are there, implicitly, because there are lines lying in those planes, and we tend to point to them…



But I would be willing to *bet* you that your Multivariable Calculus class did not go through an extremely detailed and rigorous discussion of unions and intersections of point sets and the construction of rays as geometric objects. I would also be willing to bet that you didn't feel the lack at the time!

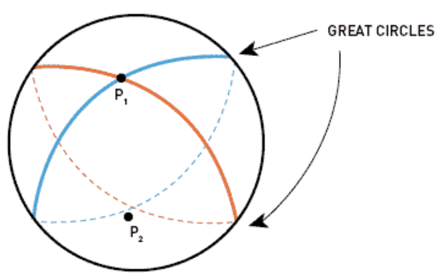
Coming back around, we all get that if you intersect two lines in a 2D Euclidean plane:



You don't call the resulting thing "an angle." You say that FOUR angles are formed by the intersection, and it's easy to describe them in a world that has rays that conveniently line up with your lines. The four angles (point sets) formed in the image above are all angles with their vertex at E :

$$\angle DEA = \overrightarrow{ED} \cup \overrightarrow{EA},\ \angle AEB = \overrightarrow{EA} \cup \overrightarrow{EB},\ \angle BEC = \overrightarrow{EB} \cup \overrightarrow{EC},\ \angle CED = \overrightarrow{EC} \cup \overrightarrow{ED}$$

So when we come around to spherical geometry, intuition says that if we intersect two lines, we're going to form four angles at the vertex. You can see the four angles with their vertex at $P1$, just like in the plane situation – the "X" shape is there, just curved over the surface of the sphere.



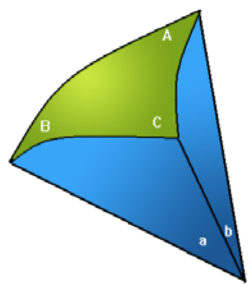
To accurately and precisely describe the geometric objects that make up those angles, taking into account the fact that "rays" have jumped the tracks, we will oh LOOK, it's RAINBOW BUTTERFLY UNICORN KITTEN!

Not kidding – the definition of "spherical angle" that you'll find, even on math websites and texts, is something along the lines of

"the angle between two intersecting arcs of great circles of a sphere measured by the plane angle formed by the tangents to the arcs at the point of intersection"

And totally glosses over the bits about "um, which angle are we talking about?" and "in terms of a point set, precisely how do I 'stop' so that I'm not simultaneously talking about both sides of the "X" at the same time?" Intuition is fine here – it still makes sense to talk about measuring the "gap".

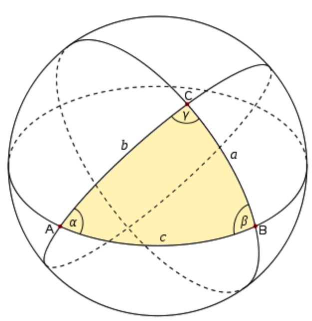
And that part is easy to describe: ***to measure an angle in spherical geometry, you just measure the angle between the two planes that were used to cut the slices that defined the great circles that created the intersecting lines.***



See the slices? Those are planes. You can measure the angle between planes using the tools you develop working with vectors. The angle between those plane faces you can see (in a normal Euclidean sense) is considered to be the measure of angle C in a spherical sense!

Do I want you to do that "by hand"? Goodness no. If you've had Multivariable Calc (aka Calc III, Vector Calc, whatever you want to call it), you do have the tools to do that in your toolbox - you measure the angle between planes by measuring the angle between their normal, and, given some coordinate points, you could work out the equations of the planes, but we are in no way going to go there. All I want you to do is use the measuring tool in the applet, and have an intuitive sense of what it just measured.

And, FINALLY…a triangle is exactly what you think it is. Look, here's a triangle (formally, it's a tedious combo of intersecting and unioning bits of three great circles with each other):



In summary, spherical geometry is an interesting one, because

- It's not the *only* geometry that uses that particular variant of the parallel postulate – there's also something called "elliptic geometry" that we're not even going to touch!
- It's got an extremely "natural" real world definition and applications (the routes planes take to take the shortest distance from point A to point B is, incidentally, a great circle)
- But defining and building it is unusual - at least compared to how we have been defining and building things! You have to imagine it floating in 3D Euclidean space, and define features of it by pointing at things that live in Euclidean space. You measure things in it by pointing at things you can measure in Euclidean 3D space. It's not this neat, self-contained thing.
- And since it relies on so many things outside itself to define itself, it's a mess *as an axiomatic system*. You can find people arguing about whether or not it should truly be considered as a standalone non-Euclidean geometry!

And, since we're in the business of studying axiomatic systems, we're not going to mess with it ☺

But there is some further reading posted if you're interested!