Suggested problems - solutions

Quadrilaterals

Material for this section references College Geometry: A Discovery Approach, 2/e, David C. Kay, Addison Wesley, 2001. In particular, see section 3.7, pp 190-193. The problems are all from section 3.7.

Problems: 1, 2, 3, 5, 7, 15, 19

#1: The first result for convex quadrilaterals is that if the quadrilateral is convex, then the diagonals intersect at an interior point on each diagonal. Since the diagonals do not intersect at an interior point, this quadrilateral cannot be convex.

This quadrilateral also fails the other criteria as well, but these are addressed in problems #2 and #3.

#2: The second result for convex quadrilaterals is that if $\diamond ABCD$ is convex, then $D$ lies in the interior of $\angle ABC$, and similar. This question shows that the vertices do not all lie in the interior of the opposite angles (and therefore the quadrilateral is not convex). (Incidentally, the only way to answer these (#1, 2, and 3) is “by looks” - you have to rely that the quadrilateral is drawn to indicate the relative locations of the vertices. I’m not thrilled by that.)

(a) $A$ is not in the interior of $\angle BCD$. 
(b) $B$ is not in the interior of $\angle ADC$.

(c) $C$ is not in the interior of $\angle BAD$.

#3: (a) $m\angle ABC = m\angle ABD + m\angle DBC$. TRUE. $D$ appears to lie in the interior of $\angle ABC$. 
(b) $\angle BAD = \angle BAC + \angle CAD$. FALSE. $C$ does not appear to lie in the interior of $\angle BAD$. It appears that $D$ is in the interior of $\angle BAC$, and that

$$\angle BAC = \angle BAD + \angle CAD$$

and so

$$\angle BAD = \angle BAC - \angle CAD$$

instead.

#5: You can prove this from the SAS hypothesis by splitting into triangles, but doing so is reduplicating the more general proof in the lecture that the angle sum of a convex quadrilateral is less than or equal to 360.

<table>
<thead>
<tr>
<th>Statement</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\triangle PQRS$ is a convex quadrilateral</td>
<td>Given.</td>
</tr>
<tr>
<td>$\angle P + \angle Q + \angle S + \angle R \leq 360$</td>
<td>Angle sum of quadrilateral (proven theorem).</td>
</tr>
<tr>
<td>$\angle R = 50, \angle S = 150$</td>
<td>Given.</td>
</tr>
<tr>
<td>$\angle Q$ is a right angle</td>
<td>Given.</td>
</tr>
<tr>
<td>$\angle Q = 90$</td>
<td>Def of right angle.</td>
</tr>
<tr>
<td>$\angle P + 90 + 150 + 50 \leq 360$</td>
<td>Substitution.</td>
</tr>
<tr>
<td>$\angle P \leq 70$</td>
<td>Algebra.</td>
</tr>
</tbody>
</table>

Note that if we hadn’t already proven the general case, I would split it up as shown in the text...and prove it just like the general proof, by noting things such as $S$ lies in the interior of $\angle PQR$, and therefore $\angle 3 + \angle 4 = \angle PSR$. 
#7: SASAA:

Suppose that $\ diamond ABCD$ and $\ diamond XYZW$ are convex quadrilaterals. If under the correspondence $\ diamond ABCD \leftrightarrow \ diamond XYZW$ we have

\[
\begin{align*}
AB & \cong XY \\
BC & \cong YZ \\
\angle B & \cong \angle Y \\
\angle C & \cong \angle Z \\
\angle D & \cong \angle W
\end{align*}
\]

then we can conclude that

$\ diamond ABCD \cong \ diamond XYZW$

i.e. the remaining corresponding parts are congruent as well:

\[
\begin{align*}
CD & \cong ZW \\
DA & \cong WX \\
\angle A & \cong \angle X
\end{align*}
\]

The proof is simply a matter of drawing in some diagonals and working with the triangle congruence results (SAS hypothesis and the various theorems: ASA, SSS, AAS).
Proof:

Given convex quadrilaterals \( \diamond ABCD \) and \( \diamond XYZW \) under the correspondence \( \diamond ABCD \leftrightarrow \diamond XYZW \), by definition we have diagonals \( \overline{AC} \) and \( \overline{XZ} \) as shown. Since the quadrilaterals are convex, \( C \) is in the interior of \( \angle DAB \), \( Z \) is in the interior of \( \angle WXY \), and by the angle addition postulate, we have (angles numbered below)

\[
\begin{align*}
m\angle 1 + m\angle 2 &= m\angle A (\angle DAB) \\
m\angle 5 + m\angle 6 &= m\angle X (\angle WXY) \\
m\angle 3 + m\angle 4 &= m\angle C (\angle BCD) \\
m\angle 7 + m\angle 8 &= m\angle Z (\angle YZW)
\end{align*}
\]

Now, since it is given that

\[
\begin{align*}
\overline{AB} &\cong \overline{XY} \\
\overline{BC} &\cong \overline{YZ} \\
\angle B &\cong \angle Y
\end{align*}
\]

by the SAS hypothesis we have \( \triangle ABC \cong \triangle XYZ \):

Therefore, \( \overline{AC} \cong \overline{XZ} \), \( \angle 2 \cong \angle 6 \) (\( m\angle 2 = m\angle 6 \) by equivalence), and \( \angle 4 \cong \angle 8 \) (\( m\angle 4 = m\angle 8 \) by equivalence):
Since we were given that \( \angle C (\angle BCD) \cong \angle Z (\angle YZW) \), by making substitutions using \( m\angle 4 = m\angle 8 \), \( m\angle 3 + m\angle 4 = m\angle C \) and \( m\angle 7 + m\angle 8 = m\angle Z \) and solving, we obtain \( m\angle 3 = m\angle 7 \) and

\[ \angle 3 \cong \angle 7 \]

We have now established that \( \triangle ACD \cong \triangle XZW \) by the AAS postulate (all marked congruences in the diagram below were either given or previously established):

And we can now have that

\[ \overline{CD} \cong \overline{ZW} \]

\[ \overline{DA} \cong \overline{WX} \]

(two of the three things we need), and also \( \angle 1 \cong \angle 5 \) (almost but not quite the last thing).

A little more algebra finishes it off: since \( m\angle 1 = m\angle 5 \), \( m\angle 1 + m\angle 2 = m\angle A \), \( m\angle 5 + m\angle 6 = m\angle X \), and \( m\angle 2 = m\angle 6 \), substituting and solving gives \( m\angle A = m\angle X \) and

\[ \angle A \cong \angle X \]
And so

\[ \Diamond ABCD \cong \Diamond XYZW \]

The statement of SASSS is

Suppose that \( \Diamond ABCD \) and \( \Diamond XYZW \) are convex quadrilaterals. If under the correspondence \( \Diamond ABCD \leftrightarrow \Diamond XYZW \) we have

\[
\begin{align*}
AB & \cong XY \\
BC & \cong YZ \\
CD & \cong ZW \\
DA & \cong WA
\end{align*}
\]

\[ \angle B \cong \angle Y \]

then we can conclude that

\[ \Diamond ABCD \cong \Diamond XYZW \]

i.e. the remaining corresponding parts are congruent as well:

\[
\begin{align*}
\angle A & \cong \angle X \\
\angle C & \cong \angle Z \\
\angle D & \cong \angle W
\end{align*}
\]

The proof is posted as a live solution. It's exactly the same idea as the proof of SASAS (lecture) and SASAA (problem #7 above).
#19: This one is easy to construct (and show as a physical demo):
(1) form a rectangle out of sticks or straws or somethin’ by tying the corners together
(2) squish it

The resulting parallelogram obviously has sides the same lengths as the original rectangle, but
not the same angles. As a construction, you can make circles to transfer the lengths of the
sides over and draw the parallelogram on the rectangle. The short sides of the parallelogram vs.
square are congruent because they’re radii of the same circles, and the longer sides are simply
moved segments. (You can prove it all formally on your own if you wish.)