Directional derivatives - suggested problems - solutions

P1: Find the derivative of \( f(x, y) = x^2e^{3y} - 5xy^4 \) at the point \((2, 0, f(2, 0))\)

\[
\begin{align*}
  f_x &= 2xe^{3y} - 5y^4 \quad f_x(2, 0) = 4 \\
  f_y &= 3x^2e^{3y} - 20xy^3 \quad f_y(2, 0) = 12
\end{align*}
\]

\[
< f_x, f_y > = < 4, 12 >
\]

(a) in the direction of the vector \( u = < \frac{1}{2}, -\frac{\sqrt{3}}{2} > \). *Note the given \( u \) is in fact a unit vector - be sure to check that.

\[
D_u f = < f_x, f_y > \cdot u
\]

\[
D_u f = < 4, 12 > \cdot < \frac{1}{2}, \frac{\sqrt{3}}{2} >
\]

\[
= \frac{4}{2} + \frac{12\sqrt{3}}{2} = 2 + 6\sqrt{3}
\]

\[
D_u f = 2 + 6\sqrt{3} \approx 12.4
\]

(b) in the direction of the vector \( v = < -3, -4 > \).

Since \( v = < -3, -4 > \) is not a unit vector, normalize it:

\[
||v|| = \sqrt{9 + 16} = 5
\]

\[
u = < -\frac{3}{5}, -\frac{4}{5} >
\]

\[
D_u f = < 4, 12 > \cdot < -\frac{3}{5}, -\frac{4}{5} >
\]

\[
= -\frac{12}{5} - \frac{48}{5} = -\frac{60}{5} = -12
\]

\[
D_u f = -12
\]

(c) in the direction of \( \theta = \frac{3\pi}{4} \).

A unit vector in the direction of \( \theta = \frac{3\pi}{4} \) is given by:

\[
u = < \cos \frac{3\pi}{4}, \sin \frac{3\pi}{4} >
\]

\[
u = < -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} >
\]

\[
D_u f = < 4, 12 > \cdot < -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} > = -\frac{4\sqrt{2}}{2} + \frac{12\sqrt{2}}{2} = -2\sqrt{2} + 6\sqrt{2} = 4\sqrt{2}
\]

\[
D_u f = 4\sqrt{2} \approx 5.66
\]
You’ll see more directional derivative problems after we cover the gradient vector in the next section. COW uses the gradient vector notation for computing directional derivatives, so I’ve listed those problems with that problem set instead of this one.

P2: Let \( f(x, y) = x^4(3y^2 - 1)^3 \).

At \((2, 1, f(2, 1)):\)

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 4x^3(3y^2 - 1)^3 \\
\frac{\partial f}{\partial x}(2, 1) &= 4 \cdot 2^3(3 \cdot 1^2 - 1)^3 \\
\frac{\partial f}{\partial x}(2, 1) &= 256 \\
\frac{\partial f}{\partial y} &= x^4 \cdot 3(3y^2 - 1)^2 \cdot (6y) \\
\frac{\partial f}{\partial y}(2, 1) &= 18 \cdot 2^4 \cdot 1(3 \cdot 1^2 - 1)^2 \\
\frac{\partial f}{\partial y}(2, 1) &= 1152 \\
\end{align*}
\]

\(< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > = < 256, 1152 >\)

(a) The derivative of \( f \) in the direction of \( \mathbf{v} = <-1, 5 > \) at the point \((2, 1, f(2, 1))\). Normalize \( \mathbf{v} = <-1, 5 >:\)

\[
||\mathbf{v}|| = \sqrt{1 + 25} = \sqrt{26}
\]

\[
\mathbf{u} = \left\langle \frac{-1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \right\rangle
\]

\[
D_{\mathbf{u}} f = < 256, 1152 > \cdot \left\langle \frac{-1}{\sqrt{26}}, \frac{5}{\sqrt{26}} \right\rangle = -\frac{256}{\sqrt{26}} + \frac{5760}{\sqrt{26}} = \frac{5504}{\sqrt{26}}
\]

\[
D_{\mathbf{u}} f = \frac{5504}{\sqrt{26}} \approx 1079
\]

(b) The slope of the line tangent to the curve in this direction at this point.

\( D_{\mathbf{u}} f \) IS the slope of the line tangent to the curve at this point:

\[
\text{slope} = \frac{5504}{\sqrt{26}} \approx 1079
\]

(c) A vector tangent to the curve in this direction at this point. A vector tangent to the curve is given by \( < u_1, y_2, D_{\mathbf{u}} f >:\)

\[
\left\langle \frac{-1}{\sqrt{26}}, \frac{5}{\sqrt{26}}, \frac{5504}{\sqrt{26}} \right\rangle
\]

(d) The equation of the line tangent to the curve in this direction at this point.

To write equation of tangent, get \( f(2, 1) - 2^4(3 \cdot 1^2 - 1)^3 = 128 \) so point is \( < 2, 1, 128 >\)

Direction vector can be scalar multiple of above tangent vector - I’d use

\(< -1, 5, 5504 >\)

(and get rid of the \( \sqrt{26} \)'s) and the equation is

\(< x, y, z > = < 2, 1, 128 > + t < -1, 5, 5504 >\)
P3: Let \( f(x, y) = \frac{x + y}{2x - y} \). At (1,0, f(1,0))

\[
f_x = \frac{(2x - y) \frac{\partial}{\partial x}(x + y) - (x + y) \frac{\partial}{\partial x}(2x - y)}{(2x - y)^2}
= \frac{(2x - y)(1) - (x + y)(2)}{(2x - y)^2}
= \frac{2x - y - 2x - 2y}{(2x - y)^2}
= -\frac{3y}{(2x - y)^2}
\]

\[
f_x(1, 0) = \frac{-3(0)}{(2 \cdot 1 - 0)^2} = 0 = 0
\]

\[
f_y = \frac{(2x - y) \frac{\partial}{\partial y}(x + y) - (x + y) \frac{\partial}{\partial y}(2x - y)}{(2x - y)^2}
= \frac{(2x - y)(1) - (x + y)(-1)}{(2x - y)^2}
= \frac{2x - y + x + y}{(2x - y)^2}
= \frac{3x}{(2x - y)^2}
\]

\[
f_y(1, 0) = \frac{3(1)}{(2 \cdot 1 - 0)^2} = \frac{3}{4}
\]

\[
< f_x, f_y > = \left\langle 0, \frac{3}{4} \right\rangle
\]

(a) The derivative of \( f \) in the direction of \( \theta = -\frac{\pi}{6} \) at the point \((1, 0, f(1,0))\).

\[
\text{theta} = -\frac{\pi}{6} \Rightarrow u = \left\langle \cos \left(-\frac{\pi}{6}\right), \sin \left(-\frac{\pi}{6}\right) \right\rangle
\]

\[
u = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle
\]

\[
D_uf = \left\langle 0, \frac{3}{4} \right\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \right\rangle = 0 - \frac{3}{8}
\]

\[
D_uf = -\frac{3}{8}
\]

(b) The slope of the line tangent to the curve in this direction at this point.
Slope of tangent is \( D_uf = -\frac{3}{8} \).

(c) A vector tangent to the curve in this direction at this point. Tangent vector \( \langle u_1, u_2, D_uf \rangle \) is

\[
\left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{3}{8} \right\rangle
\]
(d) The equation of the line tangent to the curve in this direction at this point.

\[
f(1, 0) = \frac{1 + 0}{2 \cdot 1} = \frac{1}{2}
\]

Point: \( \langle 1, 0, \frac{1}{2} \rangle \)

Direction: \( \langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, -\frac{3}{8} \rangle \) OR \( \langle 4\sqrt{3}, -4, -3 \rangle \) (multiply by 8)

Equation: \( (x, y, z) = \langle 1, 0, \frac{1}{2} \rangle + t < 4\sqrt{3}, -4, -3 > \)