Vector fields, line integrals, and Green’s Theorem

grad, div, curl

Recall that for a multivariable scalar function $f(x, y)$ [or $f(x, y, z)$ - unless otherwise stated, figure that anything we develop with 2 variables extends to 3 or more], the gradient function of $f$ is defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

Example:

For $f(x, y) = x^2y^3$, find $\nabla f(x, y)$.

The point to the preceding is that the gradient of a scalar function always produces a vector field:

$$\mathbf{F}(x, y) = \nabla f(x, y) = f_x(x, y)i + f_y(x, y)j$$

One question that’s natural to ask is does that work in the other direction: is every vector field the gradient of some scalar function? The answer is “no, not always, but sometimes,” and that one will be addressed in the material for conservative vector fields. The purpose of this lecture is different: we’re going to introduce the idea of using the gradient symbol $\nabla$ notation as a vector operator that can be applied to either a scalar function (producing the above mentioned gradient), but can also applied to vector fields using vector operations.

So this next bit is technically notation abuse, but it makes for a very simple way to express and remember three key formulas.
**The \( \nabla \) operator**

Consider \( \nabla \) to be a vector “operator”

\[
\nabla = < \frac{\partial}{\partial x}, \frac{\partial}{\partial y} >
\]

That’s where the notation abuse comes in, because \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) are not quantities or functions. \( \frac{\partial}{\partial x} \) means “take the derivative with respect to \( x \) of ...” and you’ll notice that the “of” isn’t specified!

So to make sure we’ve got this perfectly straight, note the difference between these two things:

\( \frac{\partial}{\partial x} (f) \) or \( \frac{\partial f}{\partial x} \) is a function - the partial derivative of \( f \) with respect to \( x \).

\( \frac{\partial}{\partial x} \) by itself is an operator looking for something to operate on - it stands for the “take the derivative with respect to \( x \) of...” part, but it needs to be paired with an operand before anything happens.

And now for more notation abuse...

\( \nabla = < \frac{\partial}{\partial x}, \frac{\partial}{\partial y} > \) is an operator in the shape of a vector.

So \( \nabla f \) would be notated

\[
\nabla f = < \frac{\partial}{\partial x}, \frac{\partial}{\partial y} > f
\]

Well, that’s not really a vector “times” a scalar, because \( < \frac{\partial}{\partial x}, \frac{\partial}{\partial y} > \) isn’t a thing that can be multiplied, but notationally, it certainly looks like one, so act like you’re “multiplying” the scalar \( f \) into the vector \( < \frac{\partial}{\partial x}, \frac{\partial}{\partial y} > \):

\[
\nabla f = < \frac{\partial}{\partial x} (f), \frac{\partial}{\partial y} (f) >
\]

And interpret the operators as operators - \( \frac{\partial}{\partial x} (f) \) doesn’t mean “some quantity times \( f \),” it still means “the derivative of \( f \) with respect to \( x \).” In other words \( \frac{\partial}{\partial x} (f) \) is still just \( \frac{\partial f}{\partial x} \)...otherwise known as \( f_x \), and

\[
\nabla f = < \frac{\partial}{\partial x} (f), \frac{\partial}{\partial y} (f) >= < f_x, f_y >
\]

exactly as it should be.

Seems like an awfully convoluted way to go about thinking about the gradient, but the notation makes two otherwise unwieldy new computational formulas that we apply to vector fields simple to recall.
Divergence

Suppose $F(x, y) = M(x, y)i + N(x, y)j$ is a vector field in the plane. We define the divergence of $F(x, y)$ by

$$\text{div } F(x, y) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

For a vector field in space $F(x, y, z) = M(x, y, z)i + N(x, y, z)j + P(x, y, z)k$,

$$\text{div } F(x, y, z) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Example:

Let $F(x, y, z) = x^2 z i + \ln(x^2 + y^2)j - 3xy^2z^3k$. Find div $F$.

In vector operator notation,

$$\text{div } F(x, y, z) = \nabla \cdot F$$
Curl

Suppose $F(x, y, z) = M(x, y, z)i + N(x, y, z)j + P(x, y, z)k$ is a vector field in space. We define the curl of $F(x, y, z)$ by

$$\text{curl } F = \left( \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right) i - \left( \frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right) j + \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) k$$

...which brings us to the real reason we like to think of $\nabla$ as a vector operator. Gradient and divergence are easy enough to remember in terms of differentiation, but the above formula looks like a nightmare (differentiate who with respect to what, where?!)...until you recognize that the pattern of this thing minus that thing is very familiar - it’s the same structure as a cross product.

In vector operator notation,

$$\text{curl } F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Example:

Let $F(x, y, z) = x^2zi + \ln(x^2 + y^2)j - 3xy^2z^3k$. Find curl $F$.

This is just a matter of getting the correct derivatives of the correct things:
Summary

Using the vector operator notation, \( \nabla = \left< \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right> \) or \( \left< \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right> \) as appropriate, we can succinctly state the three quantities grad, div, and curl as

- \( \text{grad } f = \nabla f \) or \( \nabla (f) \).
- \( \text{div } \mathbf{F} = \nabla \cdot \mathbf{F} \)
- \( \text{curl } \mathbf{F} = \nabla \times \mathbf{F} \)