Vector fields, line integrals, and Green's Theorem

Line integrals – suggested problems – solutions

Recall the steps are

(a) Parameterize the curve if not already done
(b) Write the function \( f(x, y) \) [or \( f(x, y, z) \)] as a function of \( t \) by composing \( f(x(t), y(t)) \) [or 
\( f(x(t), y(t), z(t)) \)].
(c) Write the expression \( ds = \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \) \( dt = \sqrt{(x'(t))^2 + (y'(t))^2} \) \( dt \)
(d) Set up the integral \( \int_C f \, ds = \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2} \) \( dt \) by making the above substitutions
(e) Integrate and evaluate

P1. Evaluate the line integral over the indicated path:
\[ \int_C (x - y) \, ds \]
\( C: \mathbf{r}(t) = 4t \mathbf{i} + 3t \mathbf{j}, \ 0 \leq t \leq 2. \)

(a) The path is already given in parameterized form; the above tells us
\( x(t) = 4t \)
\( y(t) = 3t \)
\( 0 \leq t \leq 2 \)

(b) The function \( f(x, y) = x - y \) becomes \( f(t) = 4t - 3t = t. \)

(c) \( \frac{dx}{dt} = 4, \frac{dy}{dt} = 3, \text{ and } ds = \sqrt{(3)^2 + (4)^2} \) \( dt = 5 \) \( dt \)

(d) \( \int_C (x - y) \, ds = \int_0^2 t \cdot 5 \) \( dt = \int_0^2 5t \) \( dt \)

(e) \( \int_0^2 5t \) \( dt = \frac{5}{2} \left[ t^2 \right]_0^2 = \frac{5}{2} [4] = 10 \)
P2. Evaluate the line integral over the indicated path:

\[ \int_C (x^2 + y^2 + z^2) \, ds \]

\[ C: \mathbf{r}(t) = \sin t \, \mathbf{i} + \cos t \, \mathbf{j} + 8t \, \mathbf{k}, \quad 0 \leq t \leq \frac{\pi}{2}. \]

(a) Already parameterized, with \( x(t) = \sin t \), \( y(t) = \cos t \), \( z(t) = 8t \).

(b)

\[ f(x, y, z) = x^2 + y^2 + z^2 \]

\[ f(x(t), y(t), z(t)) = \sin^2 t + \cos^2 t + 64t^2 = 1 + 64t^2 \]

(c) \( \frac{dx}{dt} = \cos t, \frac{dy}{dt} = -\sin t, \frac{dz}{dt} = 8 \), and

\[ ds = \sqrt{(\cos t)^2 + (-\sin t)^2 + 8^2} \, dt = \sqrt{1 + 64} \, dt = \sqrt{65} \, dt \]

(d), (e)

\[ \int_C (x^2 + y^2 + z^2) \, ds = \int_0^{\pi/2} (1 + 64t^2)(\sqrt{65}) \, dt = \sqrt{65} \int_0^{\pi/2} (1 + 64t^2) \, dt \]

\[ = \sqrt{65} \left[ t + \frac{64}{3} t^3 \right]_0^{\pi/2} = \sqrt{65} \left[ \frac{\pi}{2} - \frac{64}{3} \left( \frac{\pi}{2} \right)^3 \right] \approx 679 \]
P3. Write a piecewise smooth parameterization of the path $C$ described below. Then, evaluate the line integral $\int_C (x + y) \, ds$ along that path.

(a) The path starting at and returning to $(0,0)$ shown in the figure below:

\[ y = x^2 \]

The path $C$ is the union of the curves $C_1 : y = x^2$, $C_2 : y = 4$, $C_3 : x = 0$. Parameterize each.

$C_1 : \begin{align*}
x(t) &= t, \\
y(t) &= t^2, \\
0 &\leq t \leq 2 \text{ (starts at } (0,0) \text{ and ends at } (2,4))
\end{align*}$

$C_2 : \begin{align*}
x(t) &= 4 - t, \\
y(t) &= 4, \\
2 &\leq t \leq 4 \text{ (starts at } (2,4) \text{ and ends at } (0,4))
\end{align*}$

$C_3 : \begin{align*}
x(t) &= 0, \\
y(t) &= 8 - t, \\
4 &\leq t \leq 8 \text{ (starts at } (0,4) \text{ and ends at } (0,0))
\end{align*}$

Is one possible parameterization.

Along $C_1$, we have $f(x, y) = x + y \rightarrow f(x(t), y(t)) = t + t^2$,
\[ ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \sqrt{1 + 2t^2} \, dt = \sqrt{1 + 4t^2} \, dt, \text{ and} \]
\[ \int_{C_1} (x + y) \, ds = \int_0^2 (t + t^2) \sqrt{1 + 4t^2} \, dt \]

Along $C_2$, we have $f(x, y) = x + y \rightarrow f(x(t), y(t)) = 4 - t + 4 = 8 - t$,
\[ ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \sqrt{(-1)^2 + 0^2} \, dt = dt, \text{ and} \]
\[ \int_{C_2} (x + y) \, ds = \int_2^4 (8 - t) \, dt \]
Along $C_3$, we have $f(x, y) = x + y \rightarrow f(x(t), y(t)) = 0 + 8 - t = 8 - t$, 

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \sqrt{0^2 + (-1)^2} \, dt = dt,$$

and

$$\int_{C_3} (x + y) \, ds = \int_4^8 (8 - t) \, dt$$

That first integral is enough of a nuisance (part of it’s trig sub or table) that I’d just plop it into Maple, so might as well just knock out all three of them while we’re there.

\[
\int_0^2 (t + t^2) \cdot \sqrt{1 + 4 t^2} \, dt : \\
\textit{evalf}(\%) \\
\frac{167}{48} \sqrt{17} - \frac{1}{12} - \frac{1}{64} \ln(4 + \sqrt{17}) \\
14.22890845
\]

\[
\int_{-2}^4 (-t + 8) \, dt \\
10
\]

\[
\int_{-4}^8 (-t + 8) \, dt \\
8
\]

\[
\int_C (x + y) \, ds = \int_{C_1} (x + y) \, ds + \int_{C_2} (x + y) \, ds + \int_{C_3} (x + y) \, ds = 14.23 + 10 + 8 = 32.23
\]
(b) Clockwise around a circle with \( r = 4 \), starting at and returning to the point \((-4, 0)\).

The basic parameterization of a circle is \( x = r \cos(t), \ y = r \sin(t) \), but you want to tweak that as needed to get the orientation and start and end points. The usual way to get the circle running clockwise is to fiddle with the negative signs.

\[
x(t) = -4 \cos(t) \\
y(t) = 4 \sin(t) \quad 0 \leq t \leq 2\pi
\]

will start at \((-4 \cos 0, 4 \sin 0) = (-4, 0)\) and will proceed in a clockwise direction (when \( t = \frac{\pi}{2} \), it will be at the point \((0, 4)\), which is where we want it to go next).

\[
f(x, y) = x + y \rightarrow f(x(t), y(t)) = -4 \cos t + 4 \sin t
\]

\[
ds = \sqrt{(4 \sin t)^2 + (4 \cos t)^2} \, dt = \sqrt{16(\sin^2 t + \cos^2 t)} \, dt = 4 \, dt
\]

\[
\int_C (x + y)\, ds = \int_0^{2\pi} (-4 \cos t + 4 \sin t) \, 4 \, dt = 16[-\sin t - \cos t]_0^{2\pi} = 16[(0-1)-(0-1)] = 0
\]

Note that if you were interpreting as lateral surface area, this, like all integrals, really gives NET area – in this case, the function dips below the \( xy \) plane as much as it lies above, and the whole thing balances out.
(c) Counterclockwise around the same circle, starting at the same point.

Also 0 - the general principle is that same curve, opposite orientation is just going to reverse the sign of the answer, and in this particular case it’s \(-0 = 0\). For practice with parameterizing, you could get a parameterization going in the opposite direction with

\[
\begin{align*}
  x(t) &= -4 \cos(t) \quad 0 \leq t \leq 2\pi \\
  y(t) &= -4 \sin(t)
\end{align*}
\]

That would start at \((-4, 0)\) and proceed to \((0, -4)\) when \(t = \frac{\pi}{2}\).

\(ds\) would come out the same, and you’d now have

\[
\int_c (x + y) \, ds = \int_0^{2\pi} (-4 \cos t - 4 \sin t) \, dt = 16 \left[ -\sin t + \cos t \right]_0^{2\pi} = 16 \left[ (0 + 1) - (0 + 1) \right] = 0
\]
(d) Clockwise around a triangle with vertices \((0,0), (2,3),\) and \((1,4)\).

We need to get the equations of the line segments that make up the triangle, then parameterize.

Let \(C_1\) be the segment from \((0,0)\) to \((1,4)\). Slope is \(4\), equation is \(y = 4x\).

\[C_1 : x(t) = t, \ y(t) = 4t, \ 0 \leq t \leq 1 \text{ (starts at } (0,0) \text{ and ends at } (1,4) \text{)}\]

Let \(C_2\) be the segment from \((1,4)\) to \((2,3)\). Slope is \(-1\), equation is \(y - 4 = -(x - 1) \Rightarrow y = -x + 5\).

\[C_2 : x(t) = t, \ y(t) = -t + 5, \ 1 \leq t \leq 2 \text{ (starts at } (1,4) \text{ and ends at } (2,3) \text{)}\]

Let \(C_3\) be the segment from \((2,3)\) to \((0,0)\). Slope is \(\frac{3}{2}\), equation is \(y = \frac{3}{2}x\). This one you’ll need to be careful with the parameterization, since it has to pick up at \(t = 2\) and move “backwards” to \((0,0)\) as \(t\) moves “forwards.”

\[C_3 : x(t) = 4 - t, \ y(t) = \frac{3}{2}(4 - t) = 6 - \frac{3}{2}t, \ 2 \leq t \leq 4 \text{ (starts at } (2,3) \text{ and ends at } (0,0) \text{)}\]
Along $C_1$, we have $f(x, y) = x + y \rightarrow f(x(t), y(t)) = t + 4t = 5t,$

\[ ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \sqrt{1^2 + (4)^2} \, dt = \sqrt{17} \, dt, \text{ and} \]

\[ \int_{C_1} (x + y) \, ds = \int_0^1 (5t) \sqrt{17} \, dt = \frac{5\sqrt{17}}{2} \left[ t^2 \right]_0^1 = \frac{5\sqrt{17}}{2} \approx 10.3 \]

Along $C_2$, we have $f(x, y) = x + y \rightarrow f(x(t), y(t)) = t + (-t + 5) = 5,$

\[ ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \sqrt{(1)^2 + (-1)^2} \, dt = \sqrt{2} \, dt, \text{ and} \]

\[ \int_{C_2} (x + y) \, ds = \int_1^2 (5\sqrt{2}) \, dt = 5\sqrt{2} \left[ t \right]_1^2 = 5\sqrt{2} [2 - 1] = 5\sqrt{2} \approx 7.07 \]

Along $C_3$, we have $f(x, y) = x + y \rightarrow f(x(t), y(t)) = (4 - t) + \left( 6 - \frac{3}{2} t \right) = 10 - \frac{5}{2} t,$

\[ ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \sqrt{(-1)^2 + \left( -\frac{3}{2} \right)^2} \, dt = \frac{\sqrt{13}}{2} \, dt, \text{ and} \]

\[ \int_{C_3} (x + y) \, ds = \int_2^4 \left( 10 - \frac{5}{2} t \right) \frac{\sqrt{13}}{2} \, dt = \frac{\sqrt{13}}{2} \left[ 10t - \frac{5}{4} t^2 \right]_2^4 = \frac{\sqrt{13}}{2} \left[ (40 - 20) - (20 - 5) \right] = \frac{5\sqrt{13}}{2} \approx 9.01 \]

\[ \int_C (x + y) \, ds = \int_{C_1} (x + y) \, ds + \int_{C_2} (x + y) \, ds + \int_{C_3} (x + y) \, ds = 10.3 + 7.07 + 9.01 = 26.08 \]
P4: The density of a wire in the shape of the circular helix $\mathbf{r}(t) = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j} + 2t \mathbf{k}$ is given by $\rho(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2)$. Find the total mass of two turns of the wire ($0 \leq t \leq 4\pi$).

Work this one out by hand for practice, but also use Maple to sketch the curve (just to remind yourself how to plot vector valued / parameterized functions), and compute the line integral in Maple to check.

The line integral $\int_C \rho \, ds$ gives the mass of wire described by the curve $C$, so that's all we're looking at here – another line integral as the previous.

The parameterization is already given, with

\[
\begin{align*}
x(t) &= 3 \cos t \\
y(t) &= 3 \sin t \\
z(t) &= 2t
\end{align*}
\]

\[
\rho(x, y, z) = \frac{1}{2}(x^2 + y^2 + z^2) \rightarrow \rho(x(t), y(t), z(t)) = \frac{1}{2}(9 \cos^2 t + 9 \sin^2 t + 4t^2) = \frac{1}{2}(9 + 4t^2)
\]

\[
ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt = \sqrt{(-3 \sin t)^2 + (3 \cos t)^2 + (2)^2} \, dt = \sqrt{9 + 4} \, dt = \sqrt{13} \, dt
\]

\[
m = \int_C \rho \, ds = \int_0^{4\pi} \frac{1}{2}(9 + 4t^2) \sqrt{13} \, dt = \frac{\sqrt{13}}{2} \left[ 9t + \frac{4t^3}{3} \right]_0^{4\pi} = \frac{\sqrt{13}}{2} \left( 36\pi + \frac{256\pi^3}{3} \right) \approx 9947.
\]

The Maple for the integral is one of the ones I used as an example of line integrals in Maple. A plot of the wire appears below:

\[
\text{with(plots)}:
\]

\[
\text{x := } t \rightarrow 3 \cos(t) : \\
\text{y := } t \rightarrow 3 \sin(t) : \\
\text{z := } t \rightarrow 2t : \\
\text{spacecurve([x(t), y(t), z(t)], t = 0 .. 4 \cdot \Pi, axes = normal, thickness = 2});}
\]
P5: Find the area of the lateral surface over the curve $C$ in the $xy$ plane and under the surface $z = f(x, y)$.

More of the same – lateral surface area was the application used to kick off line integrals. Both of these are simple to set up and integrate...

(a) $f(x, y) = xy$, $C$ is the line segment from $(0,0)$ to $(2,4)$.

Curve $C$ has equation $y = 2x$; parameterize with

\[
\begin{align*}
x(t) &= t \\
y(t) &= 2t
\end{align*}
\]

$0 \leq t \leq 2$

\[
f(x, y) = xy \rightarrow f(x(t), y(t)) = t(2t) = 2t^2
\]

\[
ds = \sqrt{(x'(t))^2 + (y'(t))^2} \, dt = \sqrt{1^2 + 2^2} \, dt = \sqrt{5} \, dt
\]

\[
\int_C (x+y) \, ds = \int_0^2 (2t^2) \sqrt{5} \, dt = \frac{2\sqrt{5}}{3} \left[ t^3 \right]_0^2 = \frac{16\sqrt{5}}{3} \approx 11.9
\]

(b) $f(x, y) = \sin x + 1$, $C$ is the line segment from $(0,0)$ to $(2,4)$.

Same curve, so same parameterization as above.

\[
f(x, y) = \sin x + 1 \rightarrow f(x(t), y(t)) = \sin t + 1
\]

\[
\int_C (x+y) \, ds = \int_0^2 (\sin t + 1) \sqrt{5} \, dt = \sqrt{5} \left[ -\cos t + t \right]_0^2 = \sqrt{5} \left[ (-\cos 2 + 2) - (-1) \right] \approx 7.64
\]