Vector fields, line integrals, and Green’s Theorem

Vector fields

We’re already acquainted with **vector valued functions** - functions that take a single scalar \( t \) as input and produce a vector as output. For example

\[
\mathbf{r}(t) = < t^2, t, 3t - 1 > = t^2\mathbf{i} + t\mathbf{j} + (3t - 1)\mathbf{k}
\]

The easiest way to visualize vector valued functions of this type is to think of \( t \) as an unseen parameter that generates ordered triplets \((x, y, z)\). We imagine the curve being traced out over time by a moving vector that points to points along it.

You can think of these functions as “scalar in, vector out.” The above function is a mapping from \( \mathbb{R} \) to \( \mathbb{R}^3 \).

We’re also acquainted with **multivariable functions** - functions that take an ordered pair (or triplet, or higher) as input, and produce a single scalar as output. For example

\[
f(x, y) = x^2 + y^2 - 3xy
\]

We’ve been visualizing multivariable functions as points in space, where the coordinates of the \( xy \) floor determine the \( z \) coordinate of the function. Allowing \( x \) and \( y \) to vary generates a surface.

Although we haven’t really been describing them this way, multivariable functions are “vector in, scalar out;” there really isn’t any distinction between an ordered pair \((x, y)\) and the vector \(< x, y >\) that points to it. The above function is a mapping from \( \mathbb{R}^2 \) to \( \mathbb{R} \).

So you can guess what’s coming next: “vector in, vector out.” It’s easy enough to write a function that is a mapping from (say) \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \), such as

\[
\mathbf{F}(x, y) = (x^2 + y)i + (2x + y)j
\]

And easy enough to evaluate it; calculate \( \mathbf{F}(1, 3) \).
So the only thing left is what is the best way to visualize it. From a strictly algebraic perspective, it’s “vector in, vector out” - the image of the vector <1, 3> is the vector <4, 5> under this mapping. It’s just inconvenient geometry to think of both of them as vectors. It’s also inconvenient geometry to think of them as both points - you’d just have the input and output lying in the same plane with no real sense of the connection between them.

The best way to visualize is to treat the input as a point, and the output as a vector with its tail at that point. A representation of \( F(1, 3) = <4, 5> \) would be this:

Of course, (1, 3) is just one point in the domain of \( F \). And the domain of \( F \) is \( \mathbb{R}^2 = \mathbb{R} \times \mathbb{R} \), the set of all real valued ordered pairs. So each point in the plane has a vector associated with it. The overall picture of the function is a bunch of points with a bunch of vectors sticking out of them.
Example:

Generate a value table for $\mathbf{F}$ (evaluate at each of the points given).

<table>
<thead>
<tr>
<th>$(x, y)$</th>
<th>$\mathbf{F}(x, y) = (x^2 + y)i + (2x + y)j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1,3)$</td>
<td>$\mathbf{F}(1,3) = (1^2 + 3)i + (2(1) + 3)j = &lt;4, 5&gt;$</td>
</tr>
<tr>
<td>$(-1,-1)$</td>
<td></td>
</tr>
<tr>
<td>$(-2,3)$</td>
<td></td>
</tr>
<tr>
<td>$(-3,-2)$</td>
<td></td>
</tr>
<tr>
<td>$(1,-3)$</td>
<td></td>
</tr>
</tbody>
</table>

Then, plot the various points in the plane with the associated vectors.
Sketching using vectors of equal magnitude

Obviously, we can’t sketch all the vectors in a vector field (infinite number of points and so on). One approach to sketching is analogous to the idea of level curves giving a picture of a surface; we look at set of vectors where $||\mathbf{F}|| = c$ (or, to make things more convenient, $||\mathbf{F}||^2 = c^2$) for various values of $c$. Note that $||\mathbf{F}||$ is a scalar valued function, and for $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $||\mathbf{F}|| = c$ will describe a curve in the plane. All vectors of equal length will have their tails on that curve.

Example:

For the function $\mathbf{F}(x, y) = (x^2 + y)\mathbf{i} + (2x + y)\mathbf{j}$, what is the expression for the function $||\mathbf{F}(x, y)||^2$?

Plot an assortment of curves $||\mathbf{F}(x, y)||^2 = c^2$, say:

- $c = 1 \rightarrow x^4 + 2x^2y + 4x^2 + 4xy + 2y^2 = 1$
- $c = 2 \rightarrow x^4 + 2x^2y + 4x^2 + 4xy + 2y^2 = 4$
- $c = 3 \rightarrow x^4 + 2x^2y + 4x^2 + 4xy + 2y^2 = 9$

You’ll want to use plotting software for this; these are implicitly defined and not a familiar shape like an ellipse or parabola. Implicitly defined curves can be plotted in Maple, MVT, Winplot, whatever, so pick something you’re comfortable with.

Then, focus in on one of the curves (say $c = 1$) and find some points that lie on that curve.

This is a bunch of tedious scratch work, so I’ll help you out with this bit and generate a few for you. The by hand approach would just be to pick a bunch of $x$’s and solve for the matching $y$’s.
Evaluate $F(x, y) = (x^2 + y)i + (2x + y)j$ at each of those points, as before. You should be able to observe that each of the vectors does have length 1, and you can attach them to the curve $|F(x, y)|^2 = 1$.

Really, we just use software

Examples of vector fields plotted in MVT and Maple are shown in the slides. Instructions for doing so are posted. The main thing of note there is the scaling - one thing you may have noticed is that if the vectors are scaled at the same scale as the curves, they tend to take over the graph. Software generated vector fields tend to scale things down a good bit - they’re proportional to themselves (correct slope/direction) and to each other (relative lengths), but not necessarily the same scale as the underlying functions.
Domain and codomain - a note

It’s implied, but I should explicitly state this: vector fields are a subset of all possible vector-to-vector mappings. In particular, we call it a vector field when the domain and the codomain are the same space. Vector fields in the plane are mappings from $\mathbb{R}^2$ (the $(x, y)$ points) to $\mathbb{R}^2$ (the $<x, y>$ vectors), and vectors fields in space are mappings from $\mathbb{R}^3$ (the $(x, y, z)$ points) to $\mathbb{R}^3$ (the $<x, y, z>$ vectors).

We could certainly look at mappings $\mathbb{R}^n \rightarrow \mathbb{R}^m$ where $n \neq m$, but we wouldn’t call them vector fields, and we’d have a whole new set of interpretations. That’s another class...

So what do they represent?

The main physical interpretation is of the vector field lines as forces, or flow lines, or basically something vector-y acting on an object at a point. For example, if you look at the field in the previous example and visualize it as flowing water, you can chart the course of an object dropped in at any point - the vectors are describing how it will be carried along. A collection of physical examples appears as a set of notes.