Vector fields, line integrals, and Green's Theorem

Vector fields – suggested problems – solutions

P1: Give five vectors which are in the vector field

\[ \mathbf{F}(x, y, z) = x^2 \mathbf{i} - (2xy + z) \mathbf{j} + (\sin z) \mathbf{k} \]

This is totally random – just pick a bunch of \((x, y, z)\) triplets and have at it. Notice that some triplets will produce the same vectors, but they aren’t really duplicates if you go to graph them, since they’re coming out of different points. However, shoot for five distinct vectors.

<table>
<thead>
<tr>
<th>((x, y, z))</th>
<th>(\mathbf{F}(x, y, z) = x^2 \mathbf{i} - (2xy + z) \mathbf{j} + (\sin z) \mathbf{k})</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, 0, 0))</td>
<td>(\mathbf{F}(0, 0, 0) = 0^2 \mathbf{i} - (2(0)(0) + 0) \mathbf{j} + (\sin 0) \mathbf{k} = &lt;0, 0, 0&gt;)</td>
</tr>
<tr>
<td>((0, 1, 0))</td>
<td>(\mathbf{F}(0, 1, 0) = 0^2 \mathbf{i} - (2(0)(1) + 0) \mathbf{j} + (\sin 0) \mathbf{k} = &lt;0, 0, 0&gt;)</td>
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<td>((0, 0, 1))</td>
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<tr>
<td>((1, 0, 0))</td>
<td>(\mathbf{F}(1, 0, 0) = 1^2 \mathbf{i} - (2(1)(0) + 0) \mathbf{j} + (\sin 0) \mathbf{k} = &lt;1, 0, 0&gt;)</td>
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<td>(\mathbf{F}(1, 1, 1) = 1^2 \mathbf{i} - (2(1)(1) + 1) \mathbf{j} + (\sin 1) \mathbf{k} = &lt;1, -3, \sin 1&gt;)</td>
</tr>
<tr>
<td>((0, 0, \frac{\pi}{2}))</td>
<td>(\mathbf{F}(0, 0, \frac{\pi}{2}) = 0^2 \mathbf{i} - (2(0)(0) + \frac{\pi}{2}) \mathbf{j} + (\sin \frac{\pi}{2}) \mathbf{k} = &lt;0, -\frac{\pi}{2}, 1&gt;)</td>
</tr>
</tbody>
</table>

P2: For the vector field \(\mathbf{F}(x, y) = yi + 2xj\)

(a) Write the expression for \(\|\mathbf{F}(x, y)\|\).

\[ \|\mathbf{F}(x, y)\| = \sqrt{y^2 + 4x^2} \]

(b) For \(c = 1\), \(c = 2\), and \(c = 3\), write expressions for the curves \(\|\mathbf{F}(x, y)\|^2 = c^2\).

They’re all ellipses:

\[ 4x^2 + y^2 = 1 \]
\[ 4x^2 + y^2 = 4 \]
\[ 4x^2 + y^2 = 9 \]
(c) Describe and sketch the above curves. These are simple enough that you can do them by hand.

Standardizing the ellipses so you can read the major and minor axes:

\[
\frac{x^2}{(1/2)^2} + y^2 = 1 \\
\frac{x^2}{(3/2)^2} + y^2 = 1 \\
\frac{x^2}{2^2} + \frac{y^2}{2^2} = 1
\]

(d) Take the curve \( \| F(x, y) \| = c^2 \) for each of the above values of \( c \) and construct a table of various \((x, y)\) pairs that lie on that curve, and find the value of \( F \) at each of these points. Add the vectors to the curves. Do enough of them so you have a feel for what the field looks like. This is tedious, but within the scope of still doable by hand – you’ll probably see a pattern emerge that gives you an overall sense of the field.

\[4x^2 + y^2 = 1\]

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>(F(x, y) = yi + 2xj)</th>
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</thead>
<tbody>
<tr>
<td>((0,1))</td>
<td>(\langle 1, 0 \rangle)</td>
<td>(\left(\frac{1}{4}, \frac{\sqrt{3}}{2}\right))</td>
<td>(\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle)</td>
</tr>
<tr>
<td>((0,-1))</td>
<td>(\langle -1, 0 \rangle)</td>
<td>(\left(\frac{1}{4}, -\frac{\sqrt{3}}{2}\right))</td>
<td>(\langle -\frac{\sqrt{3}}{2}, \frac{1}{2} \rangle)</td>
</tr>
<tr>
<td>(\left(\frac{1}{2},0\right))</td>
<td>(\langle 0,1 \rangle)</td>
<td>(\left(-\frac{1}{4}, \frac{\sqrt{3}}{2}\right))</td>
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<td>(\langle -\frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle)</td>
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</table>
\[4x^2 + y^2 = 4\]

<table>
<thead>
<tr>
<th>(x, y)</th>
<th>(\mathbf{F}(x, y) = yi + 2xj)</th>
<th>(x, y)</th>
<th>(\mathbf{F}(x, y) = yi + 2xj)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 2)</td>
<td>(&lt;2, 0&gt;)</td>
<td>((\frac{1}{2}, \sqrt{3}))</td>
<td>(&lt;\sqrt{3}, 1&gt;)</td>
</tr>
<tr>
<td>(0, -2)</td>
<td>(&lt;-2, 0&gt;)</td>
<td>((\frac{1}{2}, -\sqrt{3}))</td>
<td>(&lt;-\sqrt{3}, 1&gt;)</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>(&lt;0, 2&gt;)</td>
<td>((\frac{1}{2}, \sqrt{3}))</td>
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</tr>
<tr>
<td>(-1, 0)</td>
<td>(&lt;0, -2&gt;)</td>
<td>((\frac{1}{2}, -\sqrt{3}))</td>
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\[4x^2 + y^2 = 9\]

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<th>(x, y)</th>
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<th>(\mathbf{F}(x, y) = yi + 2xj)</th>
</tr>
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<tbody>
<tr>
<td>(0, 3)</td>
<td>(&lt;3, 0&gt;)</td>
<td>((\sqrt{5}))</td>
<td>(&lt;\sqrt{5}, 2&gt;)</td>
</tr>
<tr>
<td>(0, -3)</td>
<td>(&lt;-3, 0&gt;)</td>
<td>((\sqrt{5}))</td>
<td>(&lt;-\sqrt{5}, 2&gt;)</td>
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<tr>
<td>(\frac{3}{2}, 0)</td>
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This is why the scaling that the software does is so useful. In this sketch, the vectors are drawn to scale, but the effect is that vectors with their tails in the second quadrant are extending down into the fourth quadrant, and vice versa. If you glanced at this one, you'd get the impression that the field was going in the opposite direction than it really is!

An object starting in QIV, for example, will be propelled up and to the left. The vectors that appear to be down and to the right in QIV are ones that originated in QII, and it's in QII that their effect takes place.
(e) Use some sort of plotting software (MVT, Maple) to produce a plot of the vector field. Recall that in the software plots, the vectors are scaled down to reduce overcrowding.

(f) Imagine that the field represents some sort of flow (water, air) and drop objects in at the points (2, 0) and (1, −3). On top of your software generated plot, chart the course the objects would take.
P3: Use MVT or Maple to plot the vector fields in the plane:

MVT shown here, Maple attached at end.

(a) \[ \mathbf{F}(x, y) = \frac{1}{8}(2xy \mathbf{i} + y^2 \mathbf{j}) \]

(b) \[ \mathbf{F}(x, y) = 15y^3 \mathbf{i} - 5xy^2 \mathbf{j} \]
P4: Use MVT or Maple to plot the vector fields in space.

MVT shown here, Maple attached at end.

\begin{align*}
\text{(a) } \mathbf{F}(x, y, z) &= \frac{x \mathbf{i} + y \mathbf{j} + z \mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} \\
\text{(b) } \mathbf{F}(x, y, z) &= \frac{yz}{y-z} \mathbf{i} + \frac{xz}{x-z} \mathbf{j} + \frac{xy}{x-y} \mathbf{k}
\end{align*}

That’s...interesting. I’m not sure what to make of it...
Vector fields suggested problems - maple plots

with(VectorCalculus):
with(Student[VectorCalculus]):
with(plots):

SetCoordinates(cartesian[x, y]):

P3(a)

\[ F := \text{VectorField} \left( \frac{1}{8} (2 \cdot x \cdot y, y^2) \right); \]

\[ \frac{1}{4} x y e_x + \frac{1}{8} y^2 e_y \]

(1)

VectorField(F, output = plot, view = [-5..5, -5..5], fieldoptions = [grid = [10, 10], thickness = 2]);
P3(b)

\[ F := \text{VectorField}\left(\left\langle 15y^3, -5xy^2 \right\rangle \right); \]

\[ 15y^3 \hat{e}_x - 5xy^2 \hat{e}_y \]

\[ \text{VectorField}(F, \text{output} = \text{plot}, \text{view} = [ -5 .. 5, -5 .. 5 ], \text{fieldoptions} = [ \text{grid} = [10, 10], \text{thickness} = 2 ]); \]

P4(a)

\[ \text{SetCoordinates}(\text{cartesian}[x, y, z]): \]

\[ F := \text{VectorField}\left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \langle x, y, z \rangle \right); \]
$$\left( \frac{x}{\sqrt{x^2+y^2+z^2}} \right) \hat{e}_x + \left( \frac{y}{\sqrt{x^2+y^2+z^2}} \right) \hat{e}_y + \left( \frac{z}{\sqrt{x^2+y^2+z^2}} \right) \hat{e}_z$$

VectorField(F, output = plot, view = [-5 .. 5, -5 .. 5, -5 .. 5], fieldoptions = [grid = [6, 6, 6], thickness = 2], axes = normal, labels = [x, y, z]);

\[ (\frac{y \cdot z}{y - z}, \frac{x \cdot z}{x - z}, \frac{x \cdot y}{x - y}) \] ;

$$\left( \frac{y \cdot z}{y - z} \right) \hat{e}_x + \left( \frac{x \cdot z}{x - z} \right) \hat{e}_y + \left( \frac{x \cdot y}{x - y} \right) \hat{e}_z$$

VectorField(F, output = plot, view = [-5 .. 5, -5 .. 5, -5 .. 5], fieldoptions = [grid = [6, 6, 6], thickness = 2], axes = normal, labels = [x, y, z]);

P4(b)
Um, yeah. I'm kinda liking MVT at the moment...