Vector spaces

The vector spaces $\mathbb{R}^n$, properties of vectors, and generalizing - introduction

Now that you have a grounding in working with vectors in concrete form, we go abstract again. Here’s the plan:

1. The set of all vectors of a given length $n$ has operations of addition and scalar multiplication on it.
2. Vector addition and scalar multiplication have certain properties (which we’ve proven already)
3. We say that the set of vectors of length $n$ with these operations form a vector space $\mathbb{R}^n$
4. So let’s take any mathematical object ($m \times n$ matrix, continuous function, discontinuous function, integrable function) and define operations that we’ll call “addition” and “scalar multiplication” on that object. If the set of those objects satisfies all the same properties that vectors do ... we’ll call them ... I know ... we’ll call them vectors, and say that they form a vector space. Functions can be vectors! Matrices can be vectors!
5. Why stop there? You know the dot product? Of two vectors? That had some properties, which were proven for vectors of length $n$? If we can define a function on our other “vectors” (matrices! continuous functions!) that has the same properties ... we’ll call it a dot product too. Actually, the more general term is inner product.

Finally, vector norms (magnitudes). Have properties. Which were proven for vectors of length $n$. So, if we can define functions on our other “vectors” that have the same properties ... those are norms, too. In fact, we can even define other norms on regular vectors in $\mathbb{R}^n$ - there is more than one way to measure the length of an vector. As long as it satisfies the properties, it’s a norm.

So, to remind you (keep this with you as we start looking at more general spaces):

A vector in $\mathbb{R}^n$ is a mathematical object that we visualize as an $n$ dimensional directed line segment: The vector $< v_1, v_2, ..., v_n >$ describes a ray with tail at the origin $< 0, 0, ..., 0 >$ and head located at $< v_1, v_2, ..., v_n >$.

Formally, the vector space $\mathbb{R}^n$ consists of a set of ordered $n$-tuples $V$, and a field $K$ (the real numbers). The elements of $V$ are the vectors, and the elements of $K$ are the scalars. Operations are defined on these objects:

- **Vector addition**: Let $u =< u_1, u_2, ..., u_n >$ and $v =< v_1, v_2, ..., v_n >$ be elements of $V$. Then $w = u + v$ is in $V$ as well and is defined by
  \[ \mathbf{w} = < w_1, w_2, ..., w_n > := < u_1 + v_1, u_2 + v_2, ..., u_n + v_n > \]

- **Scalar multiplication**: Let $u =< u_1, u_2, ..., u_n > \in V$ and $k \in K$. Then $w = ku$ is in $V$ as well and is defined by
  \[ \mathbf{w} = < w_1, w_2, ..., w_n > := < ku_1, ku_2, ..., ku_n > \]
The elements of $V$ and $K$ with these operations exhibit the following properties:

**Closure axioms:**
- $u + v \in V$. [The sum of two vectors is still a vector (of length $n$). This is true by the definition of vector addition.] We say $R^n$ is closed under addition.
- $ku \in V$. [The scalar multiple of a vector is still of vector (of length $n$). This is true by the definition of scalar multiplication.] We say $R^n$ is closed under addition.

**Addition axioms:**
- **Commutativity** of vector addition and scalar multiplication:
  
  $$u + v = v + u \quad \text{and} \quad kv = vk$$

- **Associativity** of vector addition:
  
  $$u + (v + w) = (u + v) + w$$

- An additive identity, called the zero vector:
  
  $$u + 0 = u$$

- Each element $u$ of $V$ has an additive inverse, denoted $-u$:
  
  $$u + (-u) = 0$$

**Scalar multiplication axioms**
- Scalar multiplication distributes over vector addition:
  
  $$k(u + v) = ku + kv$$

- A vector distributes over a sum of scalars:
  
  $$(k + m)u = ku + mu$$

- Scalar-vector associativity:
  
  $$k(mu) = (km)u$$

- There is a multiplicative identity, denoted 1, for the operation of scalar multiplication:
  
  $$1u = u$$

Vector spaces are also known as **linear spaces**.

And if anyone asks to to “show a set with some operations is a vector space”, it means ... you must work one by one through that list of ten properties, proving ALL of them for that set with those operations. To see this in action, there’s a live example. I’m going to provide the worked version of the notes (it’s long), but please watch the “talk-through” version as well, so I can point out what it is you’re looking for and at!
Example:
Let $V$ be the set of continuous functions on the interval $[0,1]$. Let $K$ be the real numbers. Let vector addition be defined as the usual operation of adding two functions: we add $f(x)$ and $g(x)$ by adding them pointwise -

$$h(x) = (f + g)(x) := f(x) + g(x) \text{ for each } x \in [0,1]$$

Similarly, define scalar multiplication in the usual way:

$$h(x) = (kf)(x) := k \cdot f(x) \text{ for each } x \in [0,1]$$

Do $V$ and $K$ with these operations form a vector space?

Solution:
What you have to do is check the properties. Go down the line:

1. Are continuous functions closed under addition; i.e. if you add two continuous functions, are you guaranteed to get a continuous function? More to the point, can you PROVE that? You might want to consult your freshman Calc text for the definition of continuity. (OK, I lied - there IS a little Calculus in here.)

**Proof:** Suppose $f(x)$ and $g(x)$ in $V$. By the definition of continuity,

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} g(x) = g(a)$$

for all $a$ in $[0,1]$. (At the endpoints of the domain, $x = 0$ and $x = 1$, we technically need to use a right limit $x \to a^+$ and a left limit $a \to a^-$ respectively. I’m not going to nitpick it that much.)

Consider

$$\lim_{x \to a} (f + g)(x)$$

By the definition of function addition, and properties of limits, we have

$$\lim_{x \to a} (f + g)(x) = \lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f + g)(a)$$

Since $\lim_{x \to a} (f + g)(x) = (f + g)(a)$ for all $a$ in $[0,1]$, the sum $(f + g)(x)$ is a continuous function on $[0,1]$. Therefore $V$ is closed under function addition.

2. Are continuous functions closed under scalar multiplication? Prove it!

**Proof:** Suppose everything supposed in the preceding proof (start out exactly the same way). Then

Consider

$$\lim_{x \to a} (kf)(x)$$

By the definition of scalar multiplication, and properties of limits, we have

$$\lim_{x \to a} (kf)(x) = \lim_{x \to a} [k \cdot f(x)] = k \cdot \lim_{x \to a} f(x) = k \cdot f(a) = (kf)(a)$$

Since $\lim_{x \to a} (kf)(x) = (kf)(a)$ for all $a$ in $[0,1]$, $(kf)(x)$ is a continuous function on $[0,1]$. Therefore $V$ is closed under scalar multiplication.
3. Is function addition commutative? Prove it.

**Proof:** Suppose \( f(x), \ g(x) \) in \( V \).

\[
(f + g)(x) = f(x) + g(x) \\
= g(x) + f(x) \quad \text{since } f(x) \text{ and } g(x) \text{ are real numbers} \\
= (g + f)(x)
\]

So

\[
(f + g)(x) = (g + f)(x)
\]

for all \( f(x), \ g(x) \) in \( V \).

4. Is it associative?

[This works exactly like the above.]

**Proof:** Suppose \( f(x), \ g(x), \ h(x) \) in \( V \).

\[
[(f + g) + h](x) = (f + g)(x) + h(x) \\
= [f(x) + g(x)] + h(x) \\
= f(x) + [g(x) + h(x)] \\
= f(x) + (g + h)(x) \\
= [f + (g + h)](x)
\]

So

\[
[(f + g) + h](x) = [f + (g + h)](x)
\]

for all \( f(x), \ g(x), \ h(x) \) in \( V \).

5. Is there a zero function that acts as an additive identity?

**Proof:** Let \( z(x) \) be the function such that \( z(x) = 0 \) for all \( x \) in \([0, 1]\). Note \( z(x) \) is simply a constant function, and is continuous. So \( z(x) \) is in \( V \). Now, for any \( f(x) \) in \( V \)

\[
(f + z)(x) = f(x) + z(x) = f(x) + 0 = f(x)
\]

and

\[
(z + f)(x) = z(x) + f(x) = 0 + f(x) = f(x)
\]

6. Is there a function that acts as a negative, so that \( [f + (-f)](x) = 0? \)

**Proof:** Define the function \( (-f)(x) \) by \( (-f)(x) = -f(x) \) for all \( x \) in \([0, 1]\). Since \( f(x) \) is continuous, \( (-f)(x) = -f(x) \) is continuous as well, so \( (-f)(x) \) is in \( V \). Then

\[
[f + (-f)](x) = f(x) + (-f)(x) = f(x) + [-f(x)] = 0
\]

and

\[
[(-f) + f](x) = (-f)(x) + f(x) = -f(x) + f(x) = 0
\]
7. Does a scalar distribute over the sum of continuous functions?

**Proof:** Suppose \( f(x), g(x) \) in \( V \), \( k \) scalar.

\[
[k(f + g)](x) = k[(f + g)(x)] \\
= k[f(x) + g(x)] \\
= kf(x) + kg(x) \\
= (kf)(x) + (kg)(x) \\
= [kf + kg](x)
\]

[Standard Conclusion]

8. Does a continuous function distribute over the sum of two scalars?

**Proof:** Suppose \( f(x) \) in \( V \), \( c, k \) scalar.

\[
[(c + k)f](x) = (c + k)[f(x)] \\
= cf(x) + kf(x) \\
= (cf)(x) + (kf)(x) \\
= [cf + kf](x)
\]

[Standard Conclusion]

9. Associativity with scalar multiplication?

**Proof:** Suppose \( f(x) \) in \( V \), \( c, k \) scalar.

\[
[(ck)f](x) = (ck)[f(x)] \\
= (ck)f(x) \\
= c[kf(x)] \\
= [c(kf)](x)
\]

[Standard Conclusion]

10. Scalar multiplicative identity?

**Proof:** Suppose \( f(x) \) in \( V \).

\[
(1f)(x) = 1f(x) = f(x)
\]

[Standard Conclusion]