Suggested problems - solutions

Properties involving elementary row operations

P1: Let \( A = \begin{bmatrix} 4 & -1 & 4 \\ 1 & 1 & 2 \\ 0 & 5 & -1 \end{bmatrix} \). Compute \( |A| \), and use your result to compute the determinants of the following matrices:

\[
|A| = 0 \left| \begin{array}{ccc} -1 & 4 & -5 \\ 1 & 2 & -1 \\ 4 & 1 & -1 \end{array} \right| = 0 - 5(8 - 4) - 1(4 + 1) = -5(4) - 1(5) = -25
\]

(a) \( \begin{bmatrix} 2 & 2 & 4 \\ 4 & -1 & 4 \\ 0 & 5 & -1 \end{bmatrix} \)

This matrix is obtained by doubling the second row of \( A \) (which doubles the determinant), then swapping the first and second rows (which switches the sign).

\[
\left| \begin{array}{ccc} 2 & 2 & 4 \\ 4 & -1 & 4 \\ 0 & 5 & -1 \end{array} \right| = 2(-1)|A| = -2(-25) = 50
\]

(b) \( \begin{bmatrix} 4 & 7 & 4 \\ 1 & 3 & 2 \\ 0 & 5 & -1 \end{bmatrix} \)

This matrix is obtained by performing the column operation \( 2C1 + C2 \rightarrow C2 \). Elementary operations of this type do not change the determinant, and

\[
\left| \begin{array}{ccc} 4 & 7 & 4 \\ 1 & 3 & 2 \\ 0 & 5 & -1 \end{array} \right| = |A| = -25
\]

(c) \( \begin{bmatrix} 4 & -1 & 4 \\ 1 & 1 & 2 \\ 3 & 8 & 5 \end{bmatrix} \)

This matrix is obtained by performing the row operation \( 3R2 + R3 \rightarrow R3 \). No change in the determinant.

\[
\left| \begin{array}{ccc} 4 & -1 & 4 \\ 1 & 1 & 2 \\ 3 & 8 & 5 \end{array} \right| = |A| = -25
\]
P2: Let \( A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \), with \( |A| = 3 \). Compute the determinants of the following matrices:

(a) \( \begin{bmatrix} d & a & g \\ e & b & h \\ f & c & i \end{bmatrix} \)

This matrix is obtained by transposing \( A: \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \), which has no effect on the determinant (\( |A^t| = |A| \)), and then swapping the first and second columns to obtain \( \begin{bmatrix} d & a & g \\ e & b & h \\ f & c & i \end{bmatrix} \), which changes the sign. So
\[
\left| \begin{array}{ccc} d & a & g \\ e & b & h \\ f & c & i \end{array} \right| = -|A| = -3
\]

(b) \( \begin{bmatrix} c & a & b \\ f & d & e \\ -3c + i & -3a + g & -3b + h \end{bmatrix} \)

This matrix is obtained by swapping the first and third columns of \( A: \begin{bmatrix} c & b & a \\ f & e & d \\ i & h & g \end{bmatrix} \) (one sign change), then swapping the second and third: \( \begin{bmatrix} c & a & b \\ f & d & e \\ i & g & h \end{bmatrix} \) (another sign change), then performing the row operation \( 3R1 + R3 \rightarrow R3 \) (no change in determinant at all). So
\[
\left| \begin{array}{ccc} c & a & b \\ f & d & e \\ -3c + i & -3a + g & -3b + h \end{array} \right| = (-1)(-1)|A| = |A| = 3
\]
P3: Prove

\[
\begin{vmatrix}
  a + b & c + d & e + f \\
  p & q & r \\
  u & v & w
\end{vmatrix}
= \begin{vmatrix}
  a & c & e \\
  p & q & r \\
  u & v & w
\end{vmatrix}
+ \begin{vmatrix}
  b & d & f \\
  p & q & r \\
  u & v & w
\end{vmatrix}
\]

By cofactor expansion along the top row

\[
\begin{vmatrix}
  a + b & c + d & e + f \\
  p & q & r \\
  u & v & w
\end{vmatrix}
= (a + b) \begin{vmatrix}
  q & r \\
  v & w
\end{vmatrix} - (c + d) \begin{vmatrix}
  p & r \\
  u & w
\end{vmatrix} + (e + f) \begin{vmatrix}
  p & q \\
  u & v
\end{vmatrix}
\]

\[
= \begin{vmatrix}
  a & q & r \\
  u & v & w
\end{vmatrix} + b \begin{vmatrix}
  q & r \\
  v & w
\end{vmatrix} - c \begin{vmatrix}
  p & r \\
  u & w
\end{vmatrix} - d \begin{vmatrix}
  p & r \\
  u & w
\end{vmatrix} + e \begin{vmatrix}
  p & q \\
  u & v
\end{vmatrix} + f \begin{vmatrix}
  p & q \\
  u & v
\end{vmatrix}
\]

... which is the sum of the cofactor expansions of

\[
\begin{vmatrix}
  a & c & e \\
  p & q & r \\
  u & v & w
\end{vmatrix}
\text{ and } \begin{vmatrix}
  b & d & f \\
  p & q & r \\
  u & v & w
\end{vmatrix}
\]

So

\[
\begin{vmatrix}
  a + b & c + d & e + f \\
  p & q & r \\
  u & v & w
\end{vmatrix}
= \begin{vmatrix}
  a & c & e \\
  p & q & r \\
  u & v & w
\end{vmatrix}
+ \begin{vmatrix}
  b & d & f \\
  p & q & r \\
  u & v & w
\end{vmatrix}
\]
P4: When we were looking at elimination, I’d emphasized that the row operation “add a multiple of a row to another row” is a legal move, but “add multiples of two rows to each other” is not, even though from the perspective of systems of equations, it doesn’t change the solution. The reason becomes apparent here - we’ll also be using row operations to help compute determinants, and it does matter. Try the following experiment - take the matrix

\[ A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \end{bmatrix} \]

and get a zero in the lower left corner in two ways:

(a) Transform \( A \) (and call the new matrix \( B \)) by performing the operation \( \frac{5}{2}R_1 + R_2 \to R_2 \).

(b) Transform \( A \) (and call the new matrix \( C \)) by performing the operation \( 5R_1 + 2R_2 \to R_2 \).

If \( A \) were the coefficient matrix of a system of equations (augmented with some right hand side), either of those would be a legal algebraic move, and you’d get the same solution (in fact, convince yourself of that as well - augment \( A \) with the right hand side \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \), try it both ways, and see if you get the same solution). However, it does matter when we’re looking at determinants - compute \( |A|, |B| \) and \( |C| \). Which operation left \( |A| \) unchanged, and which did not?

\[ B = \begin{bmatrix} -2 & 0 \\ 5 & \frac{29}{2} \end{bmatrix} \]

\[ C = \begin{bmatrix} -2 & 3 \\ 0 & 29 \end{bmatrix} \]

\[ |A| = -14 - 15 = -29 \]
\[ |B| = -2 \left( \frac{29}{2} \right) - 0 = -29 \]
\[ |C| = -2(29) - 0 = -58 \]

\( |B| = |A| \), but \( |C| \) does not (it has, in fact, been multiplied by a factor of 2).