Proof that $|AB| = |A| \ast |B|$

The trick is to build it up in stages, using the idea of elementary matrices.

Suppose that we use a succession of elementary row operations on $A$ to introduce zeros above and below the main diagonal. Assume for the sake of simplicity that we don’t have to do any row swaps. We can introduce these zeros using only row operations of type 3: add a multiple of a row to another row. [It’s not necessary to ever have to use the constant multiple row op - keep in mind I’m just shooting for zeros; I don’t want 1’s on the diagonals]

Each of these row operations can be represented by multiplication of $A$ by an elementary matrix of type 3, producing in the end a diagonal matrix $A'$.

$(\text{bunch of elementary matrices})A = A'$

By using inverses, we may solve for $A$. Recall that the inverse of an elementary matrix is another elementary matrix, so

$A = (\text{bunch of elementary matrices})A'$

i.e., $A$ can be expressed as the product of elementary matrices of type 3 and a matrix $A'$:

$A = E_kE_{k-1}...E_2E_1A'$

where $A'$ is the diagonal matrix resulting from applying the row operations to $A$.

We now have

$AB = E_k...E_1A'B$

Now, we need to establish a few things ... 

- The determinant of an elementary matrix of type 3 is always 1:
  
  Consider the elementary row operation of $kR_i + R_j \rightarrow R_j$ applied to the identity matrix. Because $i \neq j$, and $I_{ij} = 0$ for $i \neq j$, the elements on the diagonal of $I$ will remain unchanged. [Sketch an identity, apply a row op of type 3, and convince yourself of that. Because of the positioning, you never affect the diagonal entries. The row op that affects the diagonal is the “multiply by a constant” row op, and we’re deliberately avoiding that]. Also, note that elementary matrices of type 3 are always triangular [we see this in the elementary matrix notes]. Since the determinant of a triangular matrix is the product of the diagonal entries [proved that already], the determinant of any elementary matrix of type 3 is 1.

- For any elementary matrix of type 3, $E$, and any matrix, say $Z$,

  $|EZ| = |Z|$

  The product $EZ$ produces exactly the same matrix as if we had applied the row operation $kR_1 + R_j$ to the matrix $Z$. We have already proven that this row operation leaves the determinant unchanged. Therefore

  $|EZ| = |Z|$

- Look at what we’ve got now ...

  $|AB| = |E_kE_{k-1}...E_3E_2E_1A'B|$
  $|AB| = |E_k(E_{k-1}...E_3E_2E_1A'B)|$ [treat this as $|E_kZ|$, where $Z = E_kE_{k-1}...E_3E_2E_1A'B$]
  $|AB| = |E_{k-1}...E_3E_2E_1A'B|$
  $|AB| = |E_{k-1}(E_kE_{k-1}...E_3E_2E_1A'B)|$ [and do it again]
  $|AB| = |E_{k-2}...E_3E_2E_1A'B|$ [and again]
  $\vdots$ = $\vdots$
  $|AB| = |A'B|$
Note now that (1) we reduced the problem to the product of a diagonal matrix $A'$ with $B$, and (2) we also know that $|A'| = |A|$, since $A'$ was obtained from $A$ entirey by using row operations of type 3, which leave the determinant unchanged. [and (3), that “...” means “formally, this is an induction proof” - but I'm not throwing that into the mix, too!]

- So, all we need to prove is that

$$|DB| = |D| * |B|$$

for a diagonal matrix $D$, which is a much easier proof. Let $d_{11}, d_{22}, ... d_{nn}$ be the diagonal entries of $D$. All other entries are zero. Consider the effect of multiplying a diagonal matrix $D$ with $B$. The entries in the first row of $DB$ are simply $d_{11}$ multiplied with the entries in the first row of $B$:

$$b_{1j} = \sum_{q=1}^{n} d_{1q}b_{qj} = d_{11}b_{1j} + d_{12}b_{2j} + ... + d_{1n}b_{nj} = d_{11}b_{1j}$$

(since only $d_{11}$ is nonzero). Similarly, $b_{2j} = d_{22}b_{2j}$ [all entries in the second row of $B$ are multiplied by $d_{22}$], and so on through $b_{nj} = d_{nn}b_{nj}$

We have proven that multiplying a row of $B$ by a constant has the same effect on the determinant. So

$$|DB| = d_{11}d_{22}...d_{nn}|B|$$

But the determinant of a diagonal matrix is the product of the diagonal entries, and

$$d_{11}d_{22}...d_{nn} = |D|$$

Therefore,

$$|DB| = |D| * |B|$$

We’ve got it!

$$|AB| = |A'B| = |A'| * |B|$$

since $A'$ is diagonal. But, since $A'$ was obtained from $A$ by use of elementary row operations of type 3, $|A'| = |A|$ and

$$|AB| = |A| * |B|$$

OK, small technical detail. Most matrices can be reduced to diagonal entirely through operations of type 3. You never need to do the constant multiple op - that’s just for getting ones on the diagonal, which we don’t need here. You may in some cases have to do a row swap if you get a zero in a pivot position ... which introduces an annoying technicality - there may be a sign swap or two in there. The technical detail involves introducing a permutation matrix $P$ to keep track of the row swaps. If you go back and look at the material on elementary row operations, where I factor a matrix as $A = LU$ ... if we took that material further, we’d see that we can factor all matrices as $PA = LU$ - you may need a permuted form of the matrix.

Introducing the potential permutation matrix into the above proof is largely an annoyance, and the very few texts that bother to prove this thing at all don’t bother with it. Just note it. That’s all I’m going to do.