Determinants

The big theorem

This is the major result that comes out of the first chunk of material ... and it’s almost a letdown, because there’s nothing new here. We’ve been building up the chunks piece by piece, and the last chunk fell into place with

\[ A^{-1} \text{ exists if and only if } A \text{ is non-singular (i.e. } |A| \neq 0) \]

This one allows us to pull together all the relationships between the determinant and the matrix \( A \), when \( A \) specifically represents the coefficient matrix of a system of equations.

We can now summarize some important results about determinants, inverses, and systems of equations:

Suppose we have a system of \( n \) equations in \( n \) variables, with coefficient matrix \( A \). Then, all of the following are equivalent:

- \( A \) is non-singular.
- \( |A| \neq 0 \).
- \( A^{-1} \) exists.
- \( AX = B \) has a unique solution, which can be computed by \( X = A^{-1}B \).
- The related homogeneous system \( AX = 0 \) has only the trivial solution.

Suppose we have a system of \( n \) equations in \( n \) variables, with coefficient matrix \( A \). Then, all of the following are equivalent:

- \( A \) is singular.
- \( |A| = 0 \).
- \( A^{-1} \) does not exist.
- \( AX = B \) has either no solution, or infinitely many solutions.
- The related homogeneous system \( AX = 0 \) has infinitely many solutions.

“Equivalent” in this context means that I can start with any one of those statements and prove the others in the set. For example ...

Suppose \( A \) is singular. Then \( |A| = 0 \) (as this is simply the definition of “singular”). If \( |A| = 0 \), then \( A^{-1} \) does not exist, by theorem. If \( A^{-1} \) does not exist, then \( AX = B \) has either no solution, or infinitely many solutions (since if \( A^{-1} \) did exist, we would have \( X = A^{-1}B \), which is a unique solution). Since it is not possible for a homogeneous system to have no solutions, it must be the case that when \( B = 0 \), we have infinitely many solutions.

You could work through similar logic on the first set of equivalent statements.

The other thing to note at this point is that, as far as the practical purpose of solving systems of equations goes ... this has no impact. Row reduction through Gaussian/Gauss-Jordan elimination remains the most efficient way to solve a system of equations. Period.

You may see a section in your book on “Cramer’s Rule.” I refuse to discuss it. It is the least efficient way to solve systems of equations (using determinants), is only reasonable in practice on \( 2 \times 2 \) and \( 3 \times 3 \) matrices, it won’t distinguish between no solution and infinitely many ... and for some inexplicable reason, students love it. It’s like the only thing they remember from Linear Algebra two years later is solving systems with Cramer’s Rule. And they come back and want you to help them program it (can you tell I speak from experience?) We aren’t touching it.