Eigenvalues and eigenvectors

Defining and computing - suggested problems - solutions

For each matrix give below, find eigenvalues and eigenvectors. Give a basis and the dimension of the eigenspace for each eigenvalue.

P1:

\[ A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \]

Solve for eigenvalues:

\[ A - \lambda I = \begin{bmatrix} 5 - \lambda & 4 \\ 1 & 2 - \lambda \end{bmatrix} \]

\[ \det(A - \lambda I) = (5 - \lambda)(2 - \lambda) - (4)(1) \]

\[ = 10 - 7\lambda + \lambda^2 - 4 \]

\[ = \lambda^2 - 7\lambda + 6 \quad \text{"characteristic polynomial"} \]

\[ = (\lambda - 6)(\lambda - 1) \]

\[ \det(A - \lambda I) = 0 \text{ when } \lambda = 6, \lambda = 1. \text{ These are the eigenvalues of } A. \]

Get eigenvectors for each eigenvalue in turn by setting up the system \((A - \lambda I)x = 0:\)

\(\lambda = 6: \)

\[ \begin{bmatrix} 5 - 6 & 4 \\ 1 & 2 - 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

Let \(x_2 = s\) and so \(x_1 = -s\).

\[ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]
<−1,1> is an eigenvector of \( A \), associated with \( \lambda = 1 \). All scalars multiples of it are also eigenvectors. The set of all eigenvectors, called the eigenspace of \( A \) associated with \( \lambda = 1 \), is the space generated by \(<−1,1>\). A basis for this space is \( \{<−1,1>\} \), and has dimension 1.

\[ \lambda = 6 : \]

\[
\begin{bmatrix}
5 & 4 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-1 & 4 & 0 \\
1 & -4 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -4 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Let \( x_2 = s \) and so \( x_1 = 4s \).

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= \begin{bmatrix}
4s \\
s
\end{bmatrix}
= s \begin{bmatrix}
4 \\
1
\end{bmatrix}
\]

<4,1> is an eigenvector of \( A \), associated with \( \lambda = 6 \). All scalars multiples of it are also eigenvectors. The set of all eigenvectors, called the eigenspace of \( A \) associated with \( \lambda = 6 \), is the space generated by \(<4,1>\). A basis for this space is \( \{<4,1>\} \), and has dimension 1.

You can get eigenvalues and eigenvectors in SciLab by using the “spec” command:

\[
\text{-->A}=[5 \ 4; \ 1 \ 2];
\text{-->[evecs,evals]}=\text{spec(A)}
\]

\[
evals =
\begin{bmatrix}
6. \\
0 \\
0 & 1.
\end{bmatrix}
\]

\[
evecs =
\begin{bmatrix}
0.9701425 & -0.7071068 \\
0.2425356 & 0.7071068
\end{bmatrix}
\]

The diagonal entries of the evals matrix are the eigenvalues; the column vectors of the evecs matrix are the vectors (so \( <0.970,0.243> \) is associated with \( \lambda = 6 \), and \( <-0.707,0.707> \) with \( \lambda = 1 \). SciLab normalizes its eigenvectors into unit vectors – normalize \(<−1,1>\) for example, and you’ll get \( <\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}>> <−0.707,0.707> \).
P2:

\[ A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \]

Solve for eigenvalues:

\[ A - \lambda I = \begin{bmatrix} 1 - \lambda & -2 \\ 1 & 4 - \lambda \end{bmatrix} \]

\[
\det(A - \lambda I) = (1 - \lambda)(4 - \lambda) - (-2)(1) \\
= 4 - 5\lambda + \lambda^2 + 2 \\
= \lambda^2 - 5\lambda + 6 \\
= (\lambda - 2)(\lambda - 3)
\]

\[ \det(A - \lambda I) = 0 \text{ when } \lambda = 2, \lambda = 3. \] These are the eigenvalues of \( A \).

Get eigenvectors for each eigenvalue in turn by setting up the system \( (A - \lambda I)x = 0 \):

\[ \lambda = 2: \]

\[
\begin{bmatrix} 1 - 2 & -2 \\ 1 & 4 - 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \xrightarrow{\text{ref}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

Let \( x_2 = s \) and so \( x_i = -2s \).

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\]

\(-2, 1\) is an eigenvector of \( A \), associated with \( \lambda = 2 \). The eigenspace of \( A \) associated with \( \lambda = 2 \) is the space generated by \(-2, 1\). A basis for this space is \{\(-2, 1\)\}, and has dimension 1.
\[ \lambda = 3; \]

\[
\begin{bmatrix}
  1 - 3 & -2 \\
  1 & 4 - 3
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

\[
\begin{bmatrix}
  -2 & -2 \\
  1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & 1 & 0 \\
  0 & 0 & 0
\end{bmatrix}
\]

Let \( x_2 = s \) and so \( x_1 = -s \).

\[
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
= \begin{bmatrix}
  -s \\
  s
\end{bmatrix}
= s \begin{bmatrix}
  -1 \\
  1
\end{bmatrix}
\]

\(<-1, 1>\) is an eigenvector of \( A \), associated with \( \lambda = 3 \). The eigenspace of \( A \) associated with \( \lambda = 3 \) is the space generated by \(<-1, 1>\). A basis for this space is \( \{<-1, 1>\} \), and has dimension 1.

P3:

\[
A = \begin{bmatrix}
  1 & 2 \\
  1 & 3
\end{bmatrix}
\]

Solve for eigenvalues:

\[
A - \lambda I = \begin{bmatrix}
  1 - \lambda & 2 \\
  1 & 3 - \lambda
\end{bmatrix}
\]

\[
\det(A - \lambda I) = (1 - \lambda)(3 - \lambda) - (2)(1)
\]

\[
= 3 - 4\lambda + \lambda^2 - 2
\]

\[
= \lambda^2 - 4\lambda + 1 \quad \text{"characteristic polynomial"}
\]

This one is in here to make the point that there's no particular reason to expect these to be factorable, and that eigenvalues can be irrational. Use the quadratic formula:

\[
\lambda = \frac{4 \pm \sqrt{16 - 4}}{2} = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}
\]

And the eigenvalues of \( A \) are \( \lambda = 2 - \sqrt{3} \) and \( \lambda = 2 + \sqrt{3} \).
Be careful when solving for eigenvectors – it’s natural to want to go to decimal and round, but look at what happens if you do that:

Try $\lambda = 2 - \sqrt{3} \approx .268$:

$$
\begin{bmatrix}
1 & .268 & 2 \\
1 & 3 & -.268 \\
.732 & 2 & 0 \\
1 & 2.732 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

According to SciLab, that coefficient matrix reduces to identity

$$
\rightarrow A = [ .732 \, 2 \, 0; \, 1 \, 2.732 \, 0 ];
\rightarrow \text{rref}(A)
\text{ans} = 
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
$$

Which would make the solution $x_1 = 0, x_2 = 0$ and give an eigenvector of $<0, 0>$. But we know that this cannot be correct; by definition, eigenvectors are nonzero, and the way we derived the whole process from the outset was by requiring that $\det(A - \lambda I) = 0$, meaning that $A - \lambda I$ is a singular matrix and cannot reduce to identity. The homogeneous system must have infinitely many solutions.

There’s nothing wrong with SciLab; the matrix I entered in is in fact nonsingular. The point is that $2 - \sqrt{3}$ is not the same number as $.732$, and rounding on eigenvector problems is enough to completely mess up the solution. Stay exact

$$\lambda = 2 - \sqrt{3}$$

$$
\begin{bmatrix}
1 -(2-\sqrt{3}) & 2 \\
1 & 3 -(2-\sqrt{3})
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

$$
\begin{bmatrix}
-1+\sqrt{3} & 2 & 0 \\
1 & 1+\sqrt{3} & 0
\end{bmatrix}
\rightarrow \text{rref}
\begin{bmatrix}
1 & 1+\sqrt{3} & 0 \\
0 & 0 & 0
\end{bmatrix}
$$
It’s worth noting that Scilab has enough precision that if you enter the square roots as square roots instead of “pre-rounding,” it’ll rref correctly.

```plaintext
-->A=[-1+sqrt(3) 2 0; 1 1+sqrt(3) 0];
-->rref(A)
ans  =
    1.    2.7320508    0.
    0.     0.         0.
```

However, you should also note that I was able to rref this

\[
\begin{bmatrix}
-1+\sqrt{3} & 2 & 0 \\
1 & 1+\sqrt{3} & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1+\sqrt{3} & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

immediately without performing any algebra at all (and you should be able to as well). Since we know up front that the coefficient matrix must be singular, and this is a 2×2 system, it must be the case always that the rows are scalar multiples of each other. In an 2×2 eigenvalues problem, you’re guaranteed a bottom row of zeros, and the surviving top row can be either of the initial two rows, or any scalar multiple of them. Since the second row already had a leading 1, I knew that in rref’d form all that would happen would be that it would move to the top, and the bottom row would zero out.

Anyway, back to the eigenvector...

Let \( x_2 = s \) and so \( x_1 = -(1+\sqrt{3})s \).

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
-(1+\sqrt{3})s \\
s
\end{bmatrix} = s \begin{bmatrix}
-(1+\sqrt{3}) \\
1
\end{bmatrix}
\]

\( <-(1+\sqrt{3}),1> \) is an eigenvector of \( A \), associated with \( \lambda = 2-\sqrt{3} \). The eigenspace of \( A \) associated with \( \lambda = 2-\sqrt{3} \) is the space generated by \( <-(1+\sqrt{3}),1> \). A basis for this space is \( \{<-(1+\sqrt{3}),1>\} \), and has dimension 1.
\[ \lambda = 2 + \sqrt{3} \]

\[
\begin{bmatrix}
1 - (2 + \sqrt{3}) & 2 \\
-1 - \sqrt{3} & 1
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 3 - (2 + \sqrt{3}) \\
1 - \sqrt{3} & 0
\end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 - \sqrt{3} \\ 0 & 0 \end{bmatrix}
\]

Let \( x_2 = s \) and so \( x_1 = -(1 - \sqrt{3})s \).

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -(1 - \sqrt{3})s \\ s \end{bmatrix} = s \begin{bmatrix} -(1 - \sqrt{3}) \\ 1 \end{bmatrix}
\]

\(< -(1 - \sqrt{3}), 1 >\) is an eigenvector of \( A \), associated with \( \lambda = 2 + \sqrt{3} \). The eigenspace of \( A \) associated with \( \lambda = 2 + \sqrt{3} \) is the space generated by \(< -(1 - \sqrt{3}), 1 >\). A basis for this space is \( \{ < -(1 - \sqrt{3}), 1 > \} \), and has dimension 1.

P4:

\[
A = \begin{bmatrix} 3 & 2 & -2 \\ -3 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}
\]

Solve for eigenvalues:

\[
A - \lambda I = \begin{bmatrix} 3 - \lambda & 2 & -2 \\ -3 & -1 - \lambda & 3 \\ 1 & 2 & -\lambda \end{bmatrix}
\]

\[
\det(A - \lambda I) = (3 - \lambda) \begin{vmatrix} -1 - \lambda & 3 \\ -3 & -1 - \lambda \end{vmatrix} - 2 \begin{vmatrix} -3 & -2 \\ 1 & -\lambda \end{vmatrix} - 2 \begin{vmatrix} -3 & -1 - \lambda \\ -2 & 2 \end{vmatrix}
\]

I’m using cofactor expansion across the top row to get the determinant. This may not always be the best approach, since we’re going to end up with a cubic polynomial. How you handle that depends on what you’re using – if you have a convenient polynomial solver sitting in front of you, there’s no problem with just multiplying it all out and letting the solver solve it. If you’re going to be doing it by hand, it may
require some creative factoring, or, you might be better off doing some Gaussian elimination first to introduce some zeros into the determinant before expanding.

Keeping with the cofactor expansion though (and watch the grouping)...

\[
\det(A - \lambda I) = (3 - \lambda) \left| \begin{array}{ccc}
1 - \lambda & 3 & -2 \\ 2 & -\lambda & -2 \\ 1 & -\lambda & 1
\end{array} \right|
\]

\[
= (3 - \lambda)[(-1 - \lambda)(-\lambda) - 3(2)] - 2[(-3)(-\lambda) - (3)(1)] - 2[(-3)(2) - (-1 - \lambda)(1)]
\]

\[
= (3 - \lambda)(\lambda + \lambda^2 - 6) - 2(3\lambda - 3) - 2(-6 + 1 + \lambda)
\]

\[
= (3 - \lambda)(\lambda^2 + \lambda - 6) - 6(\lambda - 1) - 2(\lambda - 5)
\]

\[
= -\lambda^3 + 2\lambda^2 + \lambda - 2
\]

At this point, the characteristic polynomial is simplified, and it’s fine to use a solver (i.e. your calculator) to solve

\[-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0\]

and get \(\lambda = 2, \lambda = 1, \lambda = -1\). I will also note though that the above does factor by grouping:

\[-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0\]

\[-\lambda^2(\lambda - 2) + 1(\lambda - 2) = 0\]

\[(\lambda - 2)(-\lambda^2 + 1) = 0\]

\[(\lambda - 2)(-\lambda + 1)(\lambda + 1) = 0\]

Either way, \(\lambda = 2, \lambda = 1, \lambda = -1\) are the eigenvalues of \(A\).

\(\lambda = 2: \)

\[
\begin{bmatrix}
3-2 & 2 & -2 \\
-3 & -1-2 & 3 \\
1 & 2 & -2
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & -2 & 0 \\
-3 & -3 & 3 & 0 \\
1 & 2 & -2 & 0
\end{bmatrix}
\xrightarrow{rref}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

From the first row, \(x_1 = 0\). Let \(x_3 = s\) and so \(x_2 = s\).
\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} = \begin{bmatrix} 0 \\ s \\ s \\
\end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 1 \\
\end{bmatrix}
\]

\(<0,1,1>\) is an eigenvector of \(A\), associated with \(\lambda = 2\). The eigenspace of \(A\) associated with \(\lambda = 2\) is the space generated by \(<0,1,1>\). A basis for this space is \(\{<0,1,1>\}\), and has dimension 1.

\(\lambda = 1\):

\[
\begin{bmatrix}
3 & -1 & 2 & -2 \\
-3 & -1 & 3 & 0 \\
1 & 2 & -1 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
2 & 2 & -2 & 0 \\
-3 & -2 & 3 & 0 \\
1 & 2 & -1 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

From the second row, \(x_2 = 0\). Let \(x_3 = s\) and so \(x_1 = s\).

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} = \begin{bmatrix} s \\ 0 \\ s \\
\end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \\
\end{bmatrix}
\]

\(<1,0,1>\) is an eigenvector of \(A\), associated with \(\lambda = 1\). The eigenspace of \(A\) associated with \(\lambda = 1\) is the space generated by \(<1,0,1>\). A basis for this space is \(\{<1,0,1>\}\), and has dimension 1.

\(\lambda = -1\):

\[
\begin{bmatrix}
3 & -(-1) & 2 & -2 \\
-3 & -1 & -(-1) & 3 \\
1 & 2 & -(-1) & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
4 & 2 & -2 & 0 \\
-3 & 0 & 3 & 0 \\
1 & 2 & 1 & 0 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Let \(x_3 = s\) and so \(x_2 = -s\) and \(x_1 = s\).
\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = s
\begin{bmatrix}
  1 \\
  -1 \\
  1
\end{bmatrix}
\]

\(<1, -1, 1>\) is an eigenvector of \(A\), associated with \(\lambda = -1\). The eigenspace of \(A\) associated with \(\lambda = -1\) is the space generated by \(<1, -1, 1>\). A basis for this space is \(\{<1, -1, 1>\}\), and has dimension 1.

**P5:**

\[
A = \begin{bmatrix}
  1 & -2 & 2 \\
  -2 & 1 & 2 \\
  -2 & 0 & 3
\end{bmatrix}
\]

Solve for eigenvalues:

\[
A - \lambda I = \begin{bmatrix}
  1 - \lambda & -2 & 2 \\
  -2 & 1 - \lambda & 2 \\
  -2 & 0 & 3 - \lambda
\end{bmatrix}
\]

\[
\det(A - \lambda I) = (1 - \lambda) \left| \begin{array}{ccc}
  1 - \lambda & 2 \\
  -2 & 3 - \lambda
\end{array} \right| - 2 \left| \begin{array}{ccc}
  2 & 2 \\
  -2 & 3 - \lambda
\end{array} \right| + 2 \left| \begin{array}{ccc}
  2 & 1 - \lambda \\
  -2 & 0
\end{array} \right|
\]

\[
= (1 - \lambda)((1 - \lambda)(3 - \lambda) - 2(0)) + 2((-2)(3 - \lambda) - (2)(-2)) + 2((-2)(0) - (1 - \lambda)(-2))
\]

\[
= (1 - \lambda)((1 - \lambda)(3 - \lambda)) + 2(-6 + 2\lambda + 4) + 2[2(1 - \lambda)]
\]

\[
= (1 - \lambda)(1 - \lambda)(3 - \lambda) + 4(1 - \lambda) + 4(1 - \lambda)
\]

\[
= (1 - \lambda)(1 - \lambda)(3 - \lambda)
\]

Notice with this one I did some creative grouping. It would also be fine to multiply the whole thing out and use a solver – if you do, you’ll be solving

\[-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0\]

It’s just worth noting that if you do multiply it all the way out, there’s no way you’ll be able to factor it.

Either way, you’ll get \(\lambda = 3\), \(\lambda = 1\), \(\lambda = 1\) as the eigenvalues of \(A\).
$\lambda = 3$:

\[
\begin{bmatrix}
1 & -3 & -2 & 2 \\
-2 & 1 & -3 & 2 \\
-2 & 0 & 3 & -3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-2 & -2 & 2 & 0 \\
-2 & -2 & 2 & 0 \\
-2 & 0 & 0 & 0
\end{bmatrix}
\xrightarrow{\text{rref}}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

From the first row, $x_1 = 0$. Let $x_3 = s$ and so $x_2 = s$.

$<0,1,1>$ is an eigenvector of $A$, associated with $\lambda = 3$. The eigenspace of $A$ associated with $\lambda = 3$ is the space generated by $<0,1,1>$. A basis for this space is $\{<0,1,1>\}$, and has dimension 1.

$\lambda = 1$:

\[
\begin{bmatrix}
1 & -1 & -2 & 2 \\
-2 & 1 & -1 & 2 \\
-2 & 0 & 3 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -2 & 2 & 0 \\
-2 & 0 & 2 & 0 \\
-2 & 0 & 2 & 0
\end{bmatrix}
\xrightarrow{\text{rref}}
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Let $x_3 = s$ and so $x_2 = s$ and $x_1 = s$.

$<1,1,1>$ is an eigenvector of $A$, associated with $\lambda = 1$. The eigenspace of $A$ associated with $\lambda = 1$ is the space generated by $<1,1,1>$. A basis for this space is $\{<1,1,1>\}$, and has dimension 1.
P6:

\[ A = \begin{bmatrix} 4 & -1 & -2 \\ 1 & 2 & -2 \\ -2 & 2 & 7 \end{bmatrix} \]

Solve for eigenvalues:

\[ A - \lambda I = \begin{bmatrix} 4 - \lambda & -1 & -2 \\ 1 & 2 - \lambda & -2 \\ -2 & 2 & 7 - \lambda \end{bmatrix} \]

\[ \det(A - \lambda I) = (4 - \lambda) \begin{vmatrix} 2 - \lambda & -2 \\ 2 & 7 - \lambda \end{vmatrix} - (-1) \begin{vmatrix} 1 & -2 \\ -2 & 7 - \lambda \end{vmatrix} - (2 - \lambda) \begin{vmatrix} 1 & -2 \\ -2 & 2 \end{vmatrix} \]

\[ = (4 - \lambda)(2(7 - \lambda) - 4) + 1[(7 - \lambda) - 4] - 2[2 - (2 - \lambda)(-2)] \]

\[ = (4 - \lambda)(14 - 9 \lambda + \lambda^2 + 4) + 3 - \lambda - 2[2 + 4 - 2 \lambda] \]

\[ = (4 - \lambda)(\lambda^2 - 9 \lambda + 18) + 3 - \lambda - 2(6 - 2 \lambda) \]

\[ = 4\lambda^2 - 36\lambda + 72 - \lambda^3 + 9\lambda^2 - 18\lambda + 3 - \lambda - 12 + 4\lambda \]

\[ = -\lambda^3 + 13\lambda^2 - 51\lambda + 63 \]

Using a polynomial solver gives \( \lambda = 7 \) and \( \lambda = 3 \).

\( \lambda = 3 \): 

\[ \begin{bmatrix} 4 - 3 & -1 & -2 \\ 1 & 2 - 3 & -2 \\ -2 & 2 & 7 - 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} 1 & -1 & -2 \\ 1 & -1 & -2 \\ -2 & 2 & 4 \end{bmatrix} \xrightarrow{ref} \begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

Let \( x_3 = t \) and \( x_2 = s \) so \( x_1 = s + 2t \). 

\[ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s + 2t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \]
<1,1,0> and <2,0,1> are eigenvectors of \( A \), associated with \( \lambda = 3 \). The eigenspace of \( A \) associated with \( \lambda = 3 \) is the space generated by \(<1,1,0> \) and \(<2,0,1> \). A basis for this space is \{<1,1,0>, <2,0,1>\}, and has dimension 2.

\[ \lambda = 7: \]

\[
\begin{bmatrix}
4 & -7 & -1 & -2 \\
1 & 2 & -7 & -2 \\
-2 & 2 & 7 & 7
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
-3 & -1 & -2 & 0 \\
1 & -5 & -2 & 0 \\
-2 & 2 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & .5 & 0 \\
0 & 1 & .5 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Let \( x_3 = t \) and so \( x_2 = -\frac{1}{2} t \) and \( x_1 = -\frac{1}{2} t \).

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
-\frac{1}{2} t \\
-\frac{1}{2} t \\
\frac{1}{2} t
\end{bmatrix}
= t
\begin{bmatrix}
-\frac{1}{2} \\
-\frac{1}{2} \\
\frac{1}{2}
\end{bmatrix}
\]

\(<-\frac{1}{2}, -\frac{1}{2}, 1> \) is an eigenvector of \( A \), associated with \( \lambda = 7 \). You could also use \(<1,1,-2> \) if you’d like it to look cleaner. The eigenspace of \( A \) associated with \( \lambda = 7 \) is the space generated by \(<1,1,-2> \). A basis for this space is \{<1,1,-2>\}, and has dimension 1.