Eigenvalues and eigenvectors

Algebraic and geometric multiplicity

For most of the examples here, I’ll be using real valued triangular matrices, since I’ll need to work with things larger than $3 \times 3$ to get the effect I want, and I want it to be easy to spot the eigenvalues. The definitions and theorems hold for any sort of matrix, however.

Example:

Consider the matrix

$$
A = \begin{bmatrix}
2 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 \\
1 & 1 & -5 & 0 \\
-2 & 3 & 1 & 2 \\
\end{bmatrix}
$$

We know that the eigenvalues of $A$ are simply the diagonal entries, and that the characteristic polynomial $\det(A - \lambda I)$ is

$$
P(\lambda) = (\lambda - 2)(\lambda - 3)(\lambda + 5)(\lambda - 2)
$$

This polynomial has four roots (and so $A$ has four eigenvalues)

$$
\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = -5, \lambda_4 = 2
$$

but two of the eigenvalues are not distinct: $\lambda_1 = \lambda_4 = 2$.

You may recall the concept of multiplicity of roots from from PreCalc:

$$
P(\lambda) = (\lambda - 2)(\lambda - 3)(\lambda + 5)(\lambda - 2) = (\lambda - 2)^2(\lambda - 3)(\lambda + 5)
$$

and we would have called $\lambda = 2$ a double root, or a root of multiplicity 2, to indicate that the factor $(\lambda - 2)$ was squared. The other roots have multiplicity 1.
**Algebraic multiplicity**

The algebraic multiplicity of an eigenvalue is defined as the multiplicity of the corresponding root of the characteristic polynomial.

**Example:**

Write the characteristic polynomial for \( A \) and give the algebraic multiplicities of each of the eigenvalues of

\[
A = \begin{bmatrix}
-2 & 0 & 0 & 0 & 0 \\
1 & 4 & 0 & 0 & 0 \\
7 & 2 & 4 & 0 & 0 \\
-1 & 1 & 1 & 7 & 0 \\
10 & -3 & 4 & 2 & 4 \\
1 & 1 & 1 & 1 & -2
\end{bmatrix}
\]

**Example:**

Use Scilab or similar to find the eigenvalues of

\[
A = \begin{bmatrix}
363 & 833 & 273 & -16 & 49 \\
-305 & -635 & -235 & -80 & -95 \\
382 & 722 & 282 & 176 & 166 \\
-6 & -6 & 14 & 22 & 12 \\
-354 & -714 & -254 & -32 & -112
\end{bmatrix}
\]

Give the algebraic multiplicity of each eigenvalue and write the characteristic polynomial.
Now back to the eigenvectors

We saw that

\[
A = \begin{bmatrix}
2 & 0 & 0 & 0 \\
1 & 3 & 0 & 0 \\
1 & 1 & -5 & 0 \\
-2 & 3 & 1 & 2 \\
\end{bmatrix}
\]

had eigenvalues of 2, 3, and -5. Take \( \lambda = 2 \), and find a basis for the eigenspace and give its dimension. You need to do this partially "by hand": don’t just type “[evects,evals] = spec(A)” into Scilab, but set up the matrix equation \((A - \lambda I)x = 0\). You can and should use Scilab to rref that, of course.
**Geometric multiplicity**

The **geometric multiplicity** of an eigenvalue is the dimension of the basis for its associated eigenspace.

In the previous example, we saw that $\lambda = 2$ had an algebraic multiplicity of 2, since it appeared as a double root of the characteristic polynomial. However, since the associated eigenspace has basis $\{ <0,0,1> \}$ with dimension 1, the **geometric multiplicity** of $\lambda = 2$ is equal to 1.

Find the geometric multiplicities of the other eigenvalues of $A$: $\lambda = 3$ and $\lambda = -5$. 
Summary

<table>
<thead>
<tr>
<th></th>
<th>Algebraic multiplicity</th>
<th>Geometric multiplicity</th>
<th>Basis for eigenspace</th>
<th>Normalized basis vectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda = 2$</td>
<td>2</td>
<td>1</td>
<td>${ &lt;0, 0, 0, 1&gt; }$</td>
<td>${ &lt;0, 0, 0, 1&gt; }$</td>
</tr>
<tr>
<td>$\lambda = 3$</td>
<td>1</td>
<td>1</td>
<td>${ &lt;0, .32, .04, 1&gt; }$</td>
<td>${ &lt;0, .3046, .0381, .9517&gt; }$</td>
</tr>
<tr>
<td>$\lambda = -5$</td>
<td>1</td>
<td>1</td>
<td>${ &lt;0, 0, -7, 1&gt; }$</td>
<td>${ &lt;0, -9899, .1414&gt; }$</td>
</tr>
</tbody>
</table>

See the accompanying lecture for a discussion of what the Scilab looks like and how to interpret.

Example:

Here’s another one: let

$$A = \begin{bmatrix} 4 & -1 & -2 \\ 1 & 2 & -2 \\ -2 & 2 & 7 \end{bmatrix}$$

and use Scilab “[evcs, evals] = spec(A)” to get both eigenvalues and eigenvectors.

Looking at the Scilab, it appears that $\lambda = 3$ has algebraic multiplicity 2, and that there are two distinct eigenvectors associated with it: $< -.0355, .8866, -.4610 >$ and $< .8714, .4407, .2153 >$.

This suggests that both these vectors form the basis for the eigenspace, which would have dimension 2, and imply that $\lambda = 3$ has geometric multiplicity 2 as well. Eigenvectors would be in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -.0355 \\ .8666 \\ -.4610 \end{bmatrix} + t \begin{bmatrix} .8714 \\ .4407 \\ .2153 \end{bmatrix}$$

This problem was also one of the suggested problems to be worked by hand in the eigenvalue intro. If you look back there, you’ll see that $\lambda = 3$ did in fact have geometric multiplicity 2, and a basis for the eigenspace was $\{ < 1, 1, 0 >, < 2, 0, 1 > \}$. That’s worth some discussion, since
those eigenvectors aren’t scalar multiples of these eigenvectors. That can happen, so don’t be alarmed by it - in theory, if done correctly, both sets of vectors should be spanning the same two dimensional space.

The point to that whole exercise is that it’s pretty simple to pull multiplicity from Scilab, and that’s typically how you’re going to do it, but you need to know how to interpret it. You might also have noticed that in the “by hand” version, I used a poly solver and never saw the factored form, so I didn’t see the algebraic multiplicity of $\lambda = 3$, but Scilab makes it obvious. When looking at the Scilab

- Be sure to recognize zeros: anything “D-17” or “D-16” is something $\times 10^{-17}$. That’s rounding error- it’s a zero.

- Be sure to only pull independent vectors as the basis for any particular eigenspace - vectors which are scalar multiples of each other don’t count as distinct vectors.

- Don’t be alarmed if occasionally the Scilab and “by hand” versions don’t match up - as long as the multiplicities are the same, you’re probably OK.

**One more thing to worry about...**

The diagram in the lecture makes this a bit more clear - how do we know that we don’t have a situation where the eigenvectors aren’t lining up nicely with the eigenvalues, because the geometric multiplicities are pushing things out of alignment?

**Theorem**

The geometric multiplicity of an eigenvalue is always less than or equal to the algebraic multiplicity.

The proof of this one is typically deferred until later in a matrix theory course (it requires “diagonalizing” a matrix, which comes up a bit later). In our case, I’d consider it beyond the scope of an intro level class. It is, however, an established and very useful result, in that it reassures us that we’ll never have any issues interpreting computer generated eigenvectors.