Eigenvalues and eigenvectors

Eigenvalue theorems – suggested problems – solutions

P1: It is know for a square matrix $A$ that $\det(A) \neq 0$. What do you know about the eigenvalues of $A$?

None of them are zero.

P2: It is known that for a square matrix $A$, $A^{-1}$ exists. What do you know about the eigenvalues of $A$?

$A^{-1}$ exists implies $A$ is nonsingular with a nonzero determinant. None of $A$’s eigenvalues are zero.

P3: It is known that for a square matrix $A$, there are infinitely many solutions to the system $AX = 0$.

What do you know about the eigenvalues of $A$?

If the homogeneous system has infinitely many solutions, then $A$ is not invertible, and therefore singular, and therefore has at least one zero eigenvalue. The result about zero / nonzero eigenvalues is really just an addition to the “big theorem” about determinants.

P4: Suppose a nonsingular matrix $A$ has eigenvalues $\lambda_1 = -1, \lambda_2 = 4$, and $\lambda_3 = \frac{1}{3}$. What are the eigenvalues of the matrix $B = 3A^{-1} + 2A^2 - 7I$?

Apply the theorems for matrix polynomial / inverse: if $\lambda$ is an eigenvalue of $A$, then

$$3\left(\frac{1}{\lambda}\right) + 2\lambda^2 - 7$$

is an eigenvalue of $B$. So the eigenvalues of $B$ are

$$(\lambda_B)_1 = 3\left(\frac{1}{-1}\right) + 2(-1)^2 - 7 = -8$$

$$(\lambda_B)_2 = 3\left(\frac{1}{4}\right) + 2(4)^2 - 7 = \frac{103}{4}$$

$$(\lambda_B)_3 = 3\left(\frac{1}{(1/3)}\right) + 2(1/3)^2 - 7 = \frac{20}{9}$$
P5: Does the converse of the matrix polynomial eigenvalue theorem hold? In fact, consider a simpler case: suppose that $\lambda_B$ is an eigenvalue of $B = A^2$, so $Bx = A^2x = \lambda_B x$. Are all solutions to $\lambda^2 = \lambda_B$ eigenvalues of the matrix $A$?

No, and there’s a really easy counterexample – just consider the identity matrix. Let $A = I$, so $B = A^2 = I^2 = I$. An eigenvalue of $B$ (in fact, all eigenvalues of $B$) is $\lambda_B = 1$. Solutions to $\lambda^2 = 1$ are $\lambda = 1$ and $\lambda = -1$, but only $\lambda = 1$ is an eigenvalue of $A = I$.

P6: Verify that each of the matrices shown below are Hermitian (if real, translate “Hermitian” as “symmetric”). Compute their eigenvalues and observe that in all cases, the eigenvalues are real valued, even if the matrix has complex entries.

(a)

$$A = \begin{bmatrix}
1 & -2 & 3 \\
-2 & 7 & 1 \\
3 & 1 & 10
\end{bmatrix}$$

By inspection $A$ is symmetric: $A = A^T$ (a)

And eigenvalues of $A$ are, yep, real.

```matlab
-->A=[1 -2 3; -2 7 1; 3 1 10];

-->spec(A)
ans =
- 0.5525451
  7.6035161
 10.949029
```

There’s really nothing all that striking about that particular matrix – after all, we’ve seen plenty of nonsymmetric matrices that happened to have real eigenvalues (and some that haven’t), but the point worth noting is that it’s guaranteed – I could have told you the eigenvalues would be real without computing them.
By inspection, $A$ is Hermitian:

$$A = \begin{bmatrix}
1 & -2+i & 3 \\
-2-i & 7 & -4i \\
3 & 4i & 10
\end{bmatrix}$$

and

$$A' = \begin{bmatrix}
1 & -2-i & 3 \\
-2+i & 7 & 4i \\
3 & -4i & 10
\end{bmatrix}$$

so $A' = \overline{A} = A$, and we know that the eigenvalues must be real.

Which they are...

```plaintext
--> A = [1 -2-%i 3; -2+%i 7 4*%i; 3 -4*%i 10]
A =

  1.        - 2. - i      3.
- 2. + i      7.          4.i
  3.        - 4.i         10.

--> spec(A)
an =

  - 0.5245965
  4.8951218
  13.629475
```