Matrices and systems of linear equations

Classifying solutions: unique, none, infinitely many

The systems we’ve been solving have all had **unique solutions**: a single set of values for $x_1$, $x_2$, ... $x_n$. You may have noticed that the reduced row echelon form of the matrix always looks like the following (using a 3 by 3 system for example):

\[
\begin{bmatrix}
1 & 0 & 0 & x_1 \\
0 & 1 & 0 & x_2 \\
0 & 0 & 1 & x_3
\end{bmatrix}
\]

The coefficient matrix reduces to an **identity matrix**. There is a **unique solution** for each of the variables.

This is not the only possibility. Consider the following system:

\[
\begin{align*}
3x_1 + 6x_2 - 3x_3 &= 6 \\
-2x_1 - 4x_2 - 3x_3 &= -1 \\
3x_1 + 6x_2 - 2x_3 &= 10
\end{align*}
\]

Put the matrix in reduced row echelon form (use your calculator or SciLab), and write it down.

Now, turn it back into a system of equations, and interpret the result.
Next, consider the following system:

\[
\begin{align*}
  x_1 + 2x_2 - x_3 &= 3 \\
  2x_1 + 4x_2 - 2x_3 &= 6 \\
  3x_1 + 6x_2 + 2x_3 &= -1
\end{align*}
\]

Put the matrix in reduced row echelon form (use your calculator or SciLab), and write it down.

Now, turn it back into a system of equations, and interpret the result.

*It’s essential to write the solution this way, using a free parameter to write expressions for \(x_1\), \(x_2\), and \(x_3\). Simply saying ‘infinitely many solutions’ is not good enough.

So, what about this matrix shows you that there’s infinitely many solutions?
Example:

\[
\begin{align*}
  x_2 + 2x_3 &= 5 \\
  x_1 + 2x_2 + 5x_3 &= 13 \\
  x_1 + 2x_3 &= 4 \\
  x_1 + x_2 + 4x_3 &= 9
\end{align*}
\]

Put the matrix in reduced row echelon form (use your calculator or SciLab), and write it down.

Now, turn it back into a system of equations, and interpret the result.

**Summarize** the results of your observations:

You get a unique solution if:

You get no solution if:

You get infinitely many solutions if: