Calculus of multivariable functions

Limits, part 1: the intuitive approach

- In single variable Calculus, we start by introducing the idea of a limit intuitively (through graphs and tables), before turning to the formal definition. We’re going to do the same thing here.

- We think of $\lim_{x \to a} f(x) = L$ in terms of “approaching” - as $x$ values get close to $a$, function values get close to $L$.

- And we think of continuity in terms of “unbroken” - can be drawn without lifting your pencil.

- Recall that most of the functions we work with (polynomial, rational, radical, trigonometric, logarithmic, exponential) are continuous on their domains.

- And the first technique we learn for evaluating limits is direct substitution- we expect that as long as we’re in the domain of the function, that $\lim_{x \to a} f(x) = f(a)$.

- So, we go through the same process with functions of two variables(or more, but as usual, we’ll stick with two because we can graph the surfaces). Keep in mind you can approach a point $(a, b)$ from all directions in the xy plane.

- **Informal definition of limit:** We say that the limit of $f(x, y)$ as $(x, y)$ approaches $(a, b)$ is $L$ and write
  \[
  \lim_{(x,y)\to(a,b)} f(x, y) = L
  \]
  if the values of $f(x, y)$ can be made arbitrarily close to $L$ by choosing points $(x, y)$ sufficiently close to $(a, b)$.

- A function is **continuous** at $(a, b)$ if and only if
  \[
  \lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)
  \]

- A function is continuous on an open region $R$ if it continuous at every point in $R$.

- We expect all the familiar functions to be continuous on their domains, and for compositions $g(h(x, y))$ of continuous $g(x)$ and $h(x, y)$ to be continuous. If we know that the function we’re interested in is continuous on its domain, and $(a, b)$ is in the domain of the function, the limit may be evaluated by direct substitution:
  \[
  \lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)
  \]

**Example:** Find $\lim_{(x,y)\to(1,2)} x^2 y + xy^2$. 

Example: Find \( \lim_{(x,y) \to (1,2)} \frac{x^2 y + 2x}{3xy + 6} \).

Example: Find \( \lim_{(x,y) \to (-1,2)} \frac{x^2 y + 2x}{3xy + 6} \).

Examples: Where are the functions below continuous?

\[
f(x, y) = \ln(xy)
\]

\[
h(x, y) = \frac{2 - x}{\sqrt{4 - x^2 - y^2}}
\]