Limits, part 1: the intuitive approach - suggested problems - solutions

In P1 - P4, find each of the following limits. Justify your answer.

P1:

\[ \lim_{(x,y) \to (6,3)} xy \cos(x - 2y) \]

This function is a product of a polynomial \((xy)\) with a cosine - the domain is all reals \((x, y)\), and the function is continuous on its domain. Limit can be obtained by direct substitution:

\[
\lim_{(x,y) \to (6,3)} xy \cos(x - 2y) = \left( 0 \cdot \cos(0) \right) = 0
\]

P2:

\[ \lim_{(x,y) \to (1,1)} \frac{x^3 + xy^2}{x^2 + y^2} \]

This function is a rational expression with domain \(\{(x, y)| (x, y) \neq 0\}\). Since \((1, 1)\) is in the domain, and the function is continuous on its domain, the limit can be obtained by direct substitution:

\[
\lim_{(x,y) \to (1,1)} \frac{x^3 + xy^2}{x^2 + y^2} = \frac{1^3 + 1 \cdot 1^2}{1^2 + 1^2} = \frac{2}{2} = 1
\]

P3:

\[ \lim_{(x,y) \to (0,0)} \frac{x^3 + xy^2}{x^2 + y^2} \]

\((0,0)\) is not in the domain, and direct substitution gives the indeterminate form \(\frac{0}{0}\). However, we can reduce

\[
\lim_{(x,y) \to (0,0)} \frac{x^3 + xy^2}{x^2 + y^2} = \lim_{(x,y) \to (0,0)} \frac{x(x^2 + y^2)}{x^2 + y^2} = \lim_{(x,y) \to (0,0)} x = 0
\]

and therefore,

\[
\lim_{(x,y) \to (0,0)} \frac{x^3 + xy^2}{x^2 + y^2} = 0
\]
P4: (Note all of this stuff extends to functions of three or more variables.)

\[
\lim_{(x,y,z)\to(0,0,0)} e^{x^2z} \tan(y + z + 1)
\]

The exponential function is continuous for all \((x, y, z)\), and \(\tan(y + z + 1)\) is continuous as long as \(y + z + 1\) is NOT an odd multiple of \(\frac{\pi}{2}\).

So, \((0,0,0)\) is in the domain, and

\[
\lim_{(x,y,z)\to(0,0,0)} e^{x^2z} \tan(y + z + 1) = e^0 \cdot 0 \cdot \tan(0 + 0 + 1) = e^0 \tan(1) = \tan(1)
\]

In P5, explain why the limit does not exist by discussing the behavior of the function near \((0,0)\). Think about what you know about the form \(\frac{C}{0}\) from single variable Calculus. It’s still true.

P5:

\[
\lim_{(x,y)\to(0,0)} \frac{x + 1}{xy^2}
\]

\((0,0)\) is not in the domain, and attempting a direct substitution gives the form \(\frac{C}{0}\) (C constant). This is not indeterminate - it indicates unbounded behavior, as the numerator approaches 1, but the denominator becomes arbitrarily small.

\[
\lim_{(x,y)\to(0,0)} \frac{x + 1}{xy^2} \text{ does not exist}
\]

(and if you plot, you’ll see it blowing up/down around \((0,0)\))
In P6 - P8, describe and sketch the set in the xy plane on which the function is continuous.

P6:

\[ f(x, y) = \sin(y \ln x) \]

The only restriction on the domain comes from \( \ln x \) [sine is defined for all \((x, y)\)]. The domain is the set \( \{(x, y)| x > 0\} \) and the function is continuous on its domain.


P7:

\[ f(x, y) = \frac{1}{\sqrt{x^2 - y}} \]

The domain is the set \( \{(x, y)| x^2 - y > 0\} \Rightarrow \{(x, y)| y < x^2\} \) and the function is continuous on its domain.
P8:

\[ f(x, y) = \sin^{-1}(x^2 + y^2) \]

The domain of \( \sin^{-1} \theta \) is \(-1 \leq \theta \leq 1\), so we must have \(-1 \leq x^2 + y^2 \leq 1\), or \(\{(x, y)|x^2 + y^2| \leq 1\}\) (which is simply \(\{(x, y)|x^2 + y^2 \leq 1\}\) since \(x^2 + y^2\) is nonnegative)