Calculus of multivariable functions

Limits, part 3: the delta-epsilon definition

Example 1: Verifying a limit using the definition

Use the definition of the limit to verify that

\[ \lim_{(x,y) \to (1,2)} x + y = 3 \]

We need to find a \( \delta \) such that \( |f(x, y) - L| < \epsilon \) whenever \( 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \). We generally proceed by working backwards and reversing the steps (which requires a bit of caution, since we’re working with inequalities).

Start by considering

\[ |f(x, y) - L| = |x + y - 3| \]

and do a little rearranging. Since we’re trying to get to an expression that has the quantities \((x - 1)\) and \((y - 2)\), try

\[ |x + y - 3| = |(x - 1) + (y - 2)| \leq |x - 1| + |y - 2| \quad \text{(triangle inequality)} \]

So

\[ |x + y - 3| \leq \sqrt{(x - 1)^2 + (y - 2)^2} \]

Now, this is where the caution comes in - it would be nice if \( \sqrt{(x - 1)^2 + (y - 2)^2} \) were less than \( \sqrt{(x - 1)^2 + (y - 2)^2} \) ... but it isn’t. However, we can say that

\[ \sqrt{(x - 1)^2} \leq \sqrt{(x - 1)^2 + (y - 2)^2} \]

\[ \sqrt{(y - 2)^2} \leq \sqrt{(x - 1)^2 + (y - 2)^2} \]

(in both cases, we’ve simply added a non-negative term under the radical, giving a larger quantity), and therefore

\[ |x + y - 3| \leq \sqrt{(x - 1)^2 + (y - 2)^2} \]

This gives us what we want, and we can go forwards - suppose \( \delta = \frac{\epsilon}{2} \). Then \( \sqrt{(x - 1)^2 + (y - 2)^2} < \delta \) implies that \( 2\sqrt{(x - 1)^2 + (y - 2)^2} < 2(\frac{\epsilon}{2}) \), and, therefore,

\[ |x + y - 3| < \epsilon \]

This satisfies the definition of the limit: Given any \( \epsilon > 0 \), there exists a \( \delta > 0 \) (in this case, \( \delta = \frac{\epsilon}{2} \)) such that \( 0 < \sqrt{(x - 1)^2 + (y - 2)^2} < \delta \) implies \( |f(x, y) - 3| < \epsilon \).