Limits, part 5: general strategy - suggested problems - solutions

Note: Some of these require a bit of creative algebra in paths/squeezing! And be sure to inspect these in MVT, to decide on whether you’re going for an exists/doesn’t exist.

Find the limit, if it exists, or show that it doesn’t.

P1:

\[
\lim_{{(x,y) \to (0,0)}} \frac{x^3}{x^4 + 2y^4}
\]

First, observe that direct substitution gives \(\frac{0}{0}\), and there’s not obvious algebra. Second, go look at the graph - it suggests DNE. (I’m not going to include these - you’ve done plenty of MVT graphing by now)

Try paths by \(y = kx\)

\[
\lim_{{(x,y) \to (0,0)}} \frac{x^3}{x^4 + 2y^4} = \lim_{{x \to 0}} \frac{x^3}{x^4 + 2(kx)^4} = \lim_{{x \to 0}} \frac{x^3}{x^4 + 2k^4x^4} = \lim_{{x \to 0}} \frac{x^3}{x^4(1 + 2k^4)} = \lim_{{x \to 0}} \frac{1}{x(1 + 2k^4)}
\]

This limit had the form \(\frac{1}{0}\) - unbounded, and the limit does not exist. Although we expect with paths to be looking for a disagreement - this is also a conclusive result! If the limit does not even exist along lines \(kx\), then certainly the limit does not exist overall!

\[
\lim_{{(x,y) \to (0,0)}} \frac{x^3}{x^4 + 2y^4} \quad \text{Does not exist.}
\]
\[ \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} \]

Bounding hint (yes, that does give away what you’re going for): Since \((|x| - |y|)^2 \geq 0\), we have

\[
\begin{align*}
|x|^2 - 2|x||y| + |y|^2 & \geq 0 \\
|x|^2 + |y|^2 & \geq 2|x||y| \\
|x||y| & \leq \frac{1}{2} (x^2 + y^2)
\end{align*}
\]

Form is \(0 \over 0\), and nothing reduces. Graph suggests limit exist and is 0. Bound absolute distance:

\[
0 \leq \lim_{(x,y) \to (0,0)} \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| = \frac{|x||y|}{\sqrt{x^2 + y^2}}
\]

The hint tells you where to go with this: \(|x||y| \leq \frac{1}{2} (x^2 + y^2)\), so:

\[
0 \leq \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| \leq \frac{1}{2} (x^2 + y^2) = \frac{1}{2} \sqrt{x^2 + y^2}
\]

Then,

\[
\lim_{(x,y) \to (0,0)} 0 \leq \lim_{(x,y) \to (0,0)} \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| \leq \lim_{(x,y) \to (0,0)} \frac{1}{2} \sqrt{x^2 + y^2}
\]

\[
0 \leq \lim_{(x,y) \to (0,0)} \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| \leq 0
\]

and therefore,

\[
\lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = 0
\]
P3:

\[
\lim_{(x,y) \to (0,0)} \frac{3x^2 + x^2y}{x^2 + 2x^3y}
\]

Form is \(\frac{0}{0}\). Algebra? Why YES!

\[
\lim_{(x,y) \to (0,0)} \frac{3x^2 + x^2y}{x^2 + 2x^3y} = \lim_{(x,y) \to (0,0)} \frac{x^2(3 + y)}{x^2(1 + 2xy)} = \lim_{(x,y) \to (0,0)} \frac{3 + y}{1 + 2xy} = \frac{3 + 0}{1 + 2(0)(0)} = 3
\]

and

\[
\lim_{(x,y) \to (0,0)} \frac{3x^2 + x^2y}{x^2 + 2x^3y} = 3
\]

P4:

\[
\lim_{(x,y) \to (2,0)} \frac{xy - 2y}{x^2 + y^2 - 4x + 4}
\]

Algebra/Paths hint: Regroup the denominator as \(y^2 + (x^2 - 4x + 4)\), and do a little factoring (top and bottom). Then, make the substitution \(p = x - 2\). As \(x \to 2\), \(p \to 0\), and \(\lim_{(x,y) \to (2,0)} f(x, y)\) becomes \(\lim_{(p,y) \to (0,0)} f(p, y)\). Then you can consider paths of the form \(y = kp\).

Note this one’s \(\to (2, 0)\), so be sure your MUT includes \(x = 2\)! Direct sub? Nope. Algebra? Nope. You’ve got a hint. Here’s how it works...

\[
\lim_{(x,y) \to (2,0)} \frac{y(x - 2)}{x^2 + y^2 - 4x + 4} = \lim_{(x,y) \to (2,0)} \frac{y(x - 2)}{y^2 + (x - 2)^2}
\]

Let \(p = x - 2\) as \(x \to 2, p \to 0\)

\[
= \lim_{(p,y) \to (2,0)} \frac{yp}{y^2 + p^2}
\]

Now, try looking at straight lines \(y = kp\):

\[
\lim_{p \to 0} \frac{(kp)p}{(kp)^2 + p^2} = \lim_{p \to 0} \frac{kp^2}{kp^2 + p^2} = \lim_{p \to 0} \frac{kp^2}{p^2(k + 1)} = \lim_{p \to 0} \frac{k}{k + 1} = \frac{k}{k + 1}
\]

Since the limit depends on \(k\) along the paths \(y = kp\), overall:

\[
\lim_{(p,y) \to (0,0)} f(p, y) \text{ DNE}
\]

and therefore

\[
\lim_{(x,y) \to (2,0)} \frac{xy - 2y}{x^2 + y^2 - 4x + 4} \text{ Does not exist.}
\]
P5:

\[ \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^2 + 1} \]

(0, 0) is in the domain - direct substitution works! (Don’t overlook the obvious with limit problems)

\[
\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^2 + 1} = \frac{0 \times 0^2}{0^2 + 0^2 + 1} = \frac{0}{1} = 0
\]

P6:

\[ \lim_{(x,y) \to (0,1)} \frac{(y - 1) \sin(x)}{x} \]

Trig/Calc 1 hint: There’s something special about \( \lim_{x \to 0} \frac{\sin(x)}{x} \). Go look it up if you don’t recall it. (It’s a theorem somewhere in your calculus book)

\[
* \lim_{x \to 0} \frac{\sin x}{x} = 1
\]

Properties of limits are useful here...

\[
\lim_{(x,y) \to (0,1)} \frac{(y - 1) \sin x}{x} = \left[ \lim_{(x,y) \to (0,1)} (y - 1) \right] \left[ \lim_{(x,y) \to (0,1)} \frac{\sin x}{x} \right] = (1 - 1)(1) = 0(1) = 0
\]