Basis and dimension

Bases for other spaces / subspaces

Some quick notes here - I used to do this the long way (ala the textbook’s approach), going back to verifying the definitions of independence and span for spaces of matrices and functions. Since you’ve gone out of the realm of $\mathbb{R}^n$ vectors, everything becomes a by hand problem, and an exercise in gratuitous algebra.

It’s great for algebra practice, but it’s inelegant, and not the mathematician’s way. The idea of isomorphic spaces is powerful - that a bijection exists between the members of one space and the other. If that bijection exists, the structure of the spaces is preserves, and solving the problem in one space solves it in the other.

So.

The space $M_{22}$. All elements in $M_{22}$ have the form $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Define a mapping $f(A)$ to the space $\mathbb{R}^4$ of vectors $v = <v_1, v_2, v_3, v_4>$ by

$$
\begin{align*}
    f(a) &= v_1 \\
    f(b) &= v_2 \\
    f(c) &= v_3 \\
    f(d) &= v_4
\end{align*}
$$

In other words, just turn $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ into $v = <a, b, c, d>$, and solve whatever it is you’re going to solve.

Subspace of symmetric matrices in $M_{22}$? That’s matrices in the form $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$. Turn it into the subspace of $\mathbb{R}^4$ of vectors in the form $v = <a, b, b, c>$.

Example: Want a basis for that space?

$$
< a, b, c > = < a, 0, 0, a > + < 0, b, b, 0 > + < 0, 0, 0, c > = a < 1, 0, 0, 0 > + b < 0, 1, 1, 0 > + c < 0, 0, 0, 1 >
$$

The set $\{< 1, 0, 0, 1 >, < 0, 1, 1, 0 >, < 0, 0, 0, 1 >\}$ spans by definition. It’s independent by inspection (this is usually something you’d verify). So, a basis for the subspace of symmetric matrices in $M_{22}$ is (turn it back)

$$
\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}
$$

Polynomials of degree $n$ or less ($a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$) map to $\mathbb{R}^{n+1}$ ($< a_n, a_{n-1}, \ldots, a_2, a_1, a_0 >$ has length $n + 1$).

You may have noticed we aren’t talking about the basis for the space of all continuous functions anymore. You can’t map it to $\mathbb{R}^n$ ... but we don’t have the tools to cope with it in an unmapped form either. In fact, anything that we have the ability to analyze independence and span on can be mapped to $\mathbb{R}^n$.

As an interesting aside though - consider that while $\mathbb{R}^n$, or polynomials, or matrices have an infinite number of elements, they have a finite number of basis vectors.

The space of all continuous functions (on say $[0, 1]$) does not have a finite basis.

It does, however, have a basis. You can represent continuous functions with Fourier series (sines and cosines). Extremely detailed explanation here (proves all the algebraic structure on continuous functions). Not light reading.

http://www.mathreference.com/top-sw,intro.html