Basis and dimension

The span of a space; generating sets ($\mathbb{R}^n$)

We’ve been looking at spaces such as $\mathbb{R}^2$ and $\mathbb{R}^3$ and asking whether a specific set of vectors spans the space; i.e. can we make all the vectors in that space out of linear combinations of our vectors. We’ve also seen that there are sets of vectors that do not span a space - one example in the last subsection you looked at was the set 

$$<1, 3, 4>, \quad <-1, 2, 3> \quad <0, 5, 7>$$

This set

- Did not span $\mathbb{R}^3$.
- We could NOT express the vector $<1, 1, 1>$ as a linear combination of this set.
- But we COULD express the vector $<3, 4, 5>$ as a linear combination of this set.

So, what we do is turn the question around. Instead of asking

Does this set of vectors span a given space?

we ask

Given this set of vectors, what space does it span?

The answer is ... a bit unsatisfying. The space that these vectors span is the space that they span. Which is, incidentally, called the span of the vectors.

Seriously, the space spanned by a given set of vectors is the set of all possible linear combinations of those vectors. It’s called the span of the vectors, or the set generated by the vectors.

So, the set generated by the vectors

$$<1, 3, 4>, \quad <-1, 2, 3> \quad <0, 5, 7>$$

is the set of all linear combinations of

$$<1, 3, 4>, \quad <-1, 2, 3> \quad <0, 5, 7>$$

i.e., the set of all vectors in the form

$$\mathbf{v} = c_1 <1, 3, 4> + c_2 < -1, 2, 3 > + c_3 <0, 5, 7>$$

Want to construct a vector in this set? Pick some values for $c_1, c_2, c_3$. For example, let $c_1 = 1$, $c_2 = -1$ and $c_2 = 2$. Then the vector

$$\mathbf{v} = <1, 3, 4> - 1 < -1, 2, 3 > + 2 <0, 5, 7> = <2, 11, 15>$$

is in the set of all linear combinations of the given vectors.

Two questions:

- I’ve already referred to the set of linear combinations as a space. I still need to prove that. We’re guaranteed to get a subset of $\mathbb{R}^3$; what do we need to show to prove we have a subspace of $\mathbb{R}^3$?
- What would this subspace look like? It can’t be all of $\mathbb{R}^3$ (which would be a solid block of every possible 3D coordinate there is), since we already know that some vectors, like $<1, 1, 1>$, are NOT in the space.
Is the set of vectors generated by a given set always a subspace?

Suppose \( \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m \) form a set of vectors in some vector space \( V \). Suppose \( Q \) is the set of all vectors generated by those vectors; i.e. \( Q \) consists of all vectors in the form

\[
\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_m\mathbf{v}_m
\]

To show that \( Q \) is a subspace of \( V \), what do we need to check?

**CLOSURE**

And what does checking closure look like?

- For any two vectors \( \mathbf{u} \) and \( \mathbf{v} \) in the space, the sum \( \mathbf{u} + \mathbf{v} \) must be in the space.
- For any vector \( \mathbf{u} \) and a scalar \( k \), \( k\mathbf{u} \) must be in the space.

**Check the vector sum:**

Let

\[
\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_m\mathbf{v}_m \\
\mathbf{u} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + ... + d_m\mathbf{v}_m
\]

\[
\mathbf{u} + \mathbf{v} = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_m\mathbf{v}_m) + (d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + ... + d_m\mathbf{v}_m) \\
= (c_1\mathbf{v}_1 + d_1\mathbf{v}_1) + (c_2\mathbf{v}_2 + d_2\mathbf{v}_2) + ... + (c_m\mathbf{v}_m + d_m\mathbf{v}_m) \\
= (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2 + ... + (c_m + d_m)\mathbf{v}_m
\]

That last line still has the form of linear combinations of the vectors \( \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m \) - it’s a sum of scalar multiples of the vectors. So \( \mathbf{u} + \mathbf{v} \) is still in the set generated by those vectors, and \( Q \) is closed under vector addition.

**You check the scalar multiple:**

Notice the setup of this problem was very general - no reference to any specific type of vector space \( V \). So this property - that the set generated by a collection of vectors is always a subspace - holds no matter what sort of “vector” we’re talking about.
You’ve encountered generated spaces before - express as a space generated by a vector / set of vectors:

- The set of vectors in the form $< a, 2a >$

- The set of vectors in the form $< a + b, a - b, b >$

The geometry of subspaces of $R^3$, revisited

We saw this a little bit the first time we looked at subspaces. At the time, what we were getting were parametric equations of lines and planes in $R^3$, and I noted that you need a Vector Calc background to interpret those. Thinking about subspaces in terms of a span of vectors gives us another way to look at those planes and lines ¡no Vector Calc needed!¡.

Consider the set of vectors

$< 1, 1, 1 >$  $< 3, -2, 4 >$

First off, we immediately know that they don’t span all of $R^3$. Why?

Start by sketching those vectors:

So, we consider the set generated by those vectors - the set of all

$v = c_1 < 1, 1, 1 > + c_2 < 3, -2, 4 >$

Now, think about what those linear combinations look like. First off, suppose $c_1 = 0$, and $c_2$ can take on an infinite number of values. What do you get? And suppose $c_2 = 0$, and $c_1$ can take on an infinite number of values? What do you get?
Now, consider all possible vector sums - what you are doing is connecting every single point on each of those lines to every other point on each of those lines:

So, you get a plane in $\mathbb{R}^3$, and the vectors $<1, 1, 1>$ and $<3, -2, 4>$ lie in that plane.

Exercise: For the set $<1, 2, 3>$, $<4, -2, -1>$

- Give three other vectors in the subspace generated by that set.

- Sketch the subset. This will, of course, depend on your artistic ability. The key things to indicate are (1) you can (more or less) sketch the vectors in 3D, and (2) some sort of scribble indicating there’s a plane between them.

Oh, and the original question that spawned all this? What does the set spanned by $<1, 3, 4>$, $<-1, 2, 3>$, $<0, 5, 7>$ look like? It’s still a plane - turns out that one of those vectors is redundant - in fact, one vector is a linear combination of the other two. That’s the next section!