Basis and dimension

Linear combinations of vectors

Given a set of vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \), a linear combination of these vectors is obtained by any combination of vector addition and scalar multiplication:

\[ c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_n \mathbf{v}_n \]

Example: \( \mathbf{v} = \langle 12, -11, -3 \rangle \) is a linear combination of the set

\( \{ \langle 1, 2, 3 \rangle, \langle -1, 5, 4 \rangle, \langle 4, 1, 3 \rangle \} \)

We say that a vector \( \mathbf{v} \) is a linear combination of a set of vectors if there exist constants such that

\[ \mathbf{v} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \ldots + c_n \mathbf{v}_n \]

To determine if a given vector is a linear combination of a given set of vectors, we set up the equations and see if there is a solution. If the solution exists (either unique or infinitely many), it is possible to write the given vector as a linear combination of the given set.

Example: Two questions - determine and express:

Is the vector \( \mathbf{v} = \langle 4, -5, 6 \rangle \) a linear combination of the set

\( \{ \langle 1, 2, 3 \rangle, \langle 1, 1, 1 \rangle, \langle 3, 3, 2 \rangle \} \)

If so, express it as a linear combination of those vectors.
Example: Is $<4,3,8>$ a linear combination of the vectors $<-1,0,1>$, $<2,1,3>$ and $<0,1,5>$? If yes, express.

Example: Is $<4,-4,6>$ a linear combination of the vectors $<1,2,-3>$, $<2,-4,6>$ and $<-1,2,-3>$? If yes, express.
As always, the “vectors” don’t have to be vectors in $\mathbb{R}^n$; we can express a matrix as a linear combination of matrices, or a function as a linear combination of functions.

**Example:** Can the matrix $\begin{bmatrix} 4 & 1 \\ 7 & 10 \end{bmatrix}$ be expressed as a linear combination of the matrices

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & -1 \\ 2 & 3 \end{bmatrix}$

**Example:** Express $2x^2 + 8x - 26$ as a linear combination of $3x^2 + 2x - 5$ and $x^2 - x + 4$.  