Suggested problems - solutions

Norms on other spaces

P1: Consider the space of continuous functions on \([a, b]\), with inner product \(f(u(x), v(x)) = \int_a^b u(x)v(x) \, dx\). Find the expression that will give \(\|g(x)\|\) for \(g(x) = x^3\), where \(\| \cdot \|\) is the norm defined by the inner product: \(\sqrt{f(v(x), v(x))}\).

\[\|g(x)\| = \sqrt{\int_a^b [g(x)]^2 \, dx} = \sqrt{\int_a^b (x^3)^2 \, dx} = \sqrt{\int_a^b x^6 \, dx} = \sqrt{\frac{1}{7}x^7 \bigg|_a^b} = \sqrt{\frac{1}{7}(b^7 - a^7)}\]

P2: Let \(u\) and \(v\) be vectors in \(V\), where \(V\) is some vector space with an inner product \(f(u, v)\) and norm \(\|v\| = \sqrt{f(v, v)}\). Prove that \(\|u\| = \|v\|\) if and only if \(u + v\) and \(u - v\) are orthogonal.

(The point to this one is you can prove properties without having any specific formula for norm or dot product in mind.) (Makes a nice final exam takehome question.)

Suppose \(\|u\| = \|v\|\). Consider \(f(u + v, u - v)\). Use the properties of the inner product to expand that out. You need to eventually conclude that that inner product = 0.

Suppose \(u + v\) and \(u - v\) are orthogonal. Then \(f(u + v, u - v) = 0\). Expand that out; you need to eventually conclude \(\|u\| = \|v\|\).

Hint: inner product properties are the same as dot product properties. If you wrote it as \((u + v) \cdot (u - v)\), you’d recognize what to do.

P3: Define \(\|f(x)\| := \max_{0 \leq x \leq 2} |f(x)|\) on the space of continuous functions.

(a) Prove it’s a norm.

* \(\|v\| \geq 0\)

In the lecture (using \([0,1]\), but the interval is irrelevant to the proof).

* \(\|v\| = 0\) iff \(v = 0\)

Suppose \(\|f(x)\| = 0\). Then \(\max_{0 \leq x \leq 2} |f(x)| = 0\). If there existed some \(a \in [0, 2]\) such that \(f(a) \neq 0\), then the maximum value for \(|f|\) would have to be at least \(|f(a)|\): \(\|f(x)\| = \max_{0 \leq x \leq 2} |f(x)| \geq f(a) > 0\), a contradiction. So \(f(x) = 0\) for all \(x \in [0, 2]\).

Don’t forget the other direction of the iff.

* \(\|k v\| = |k||v|\)

All yours.

* \(\|v + u\| \leq \|v\| + \|u\|\)

Also yours. Note these have arguments identical to the proof for \(\|v\|_{\infty}\) on \(\mathbb{R}^n\); you mainly need to translate the notation over to this problem.
(b) Compute $||f(x)||$ for $f(x) = e^x$.

Since $f(x) = e^x$ is an increasing function ($f'(x) = e^x > 0$ for all $x$), and since $f(x) > 0$, the maximum value on $[0, 2]$ occurs at $x = 2$.

$$\max_{0 \leq x \leq 2} |f(x)| = f(2) = e^2$$

(c) Compute $||f(x)||$ for $f(x) = 5x - 7$

Since $f(x) = 5x - 7$ is linear, extrema occur at the ends of the interval: $f(0) = -7$ and $f(2) = 3$. Keep in mind you’re looking for max in absolute value:

$$\max_{0 \leq x \leq 2} |f(x)| = |f(0)| = |-7| = 7$$

P4: We don’t really have any norms defined on spaces of matrices yet. We do have one function that takes a matrix and returns a scalar, though - the determinant. Consider $|\det(A)|$ for $A \in M_{22}$. [Notation: those are absolute value bars, which is why I’m using $\det(A)$ and not $|A|$ for determinant.] Explain why this function does not work as a norm on $M_{22}$. Which properties does it pass (prove it), and which does it fail (give counterexample).

All yours. Be looking for counterexamples; it fails almost everything.