Vector spaces

Norms on $R^n$

For vectors in $R^n$, we have the magnitude of a vector given by

$$||v|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2} = \sqrt{n \sum_{i=1}^{n} (v_i)^2}$$

This is the geometric length of that vector (in however many dimensions). The symbol $||v||$ is used to indicate magnitude ... and will, like everything else in this section, have a more general definition. The formula that gives the magnitude of a vector in $R^n$ can be thought of as a function that (1) takes a vector as input, (2) returns a scalar as output, and (3) has certain properties. The more general name for this type of function is a norm, and in $R^n$, “norm” and “magnitude” tend to get used interchangeably.

In general, a norm is any function defined on a vector space $V$ that takes a vector as input and returns a scalar as output:

$$f : V \ni v \rightarrow k \in R$$

Instead of writing $f(v)$, we generally denote the norm by $||v||$. Furthermore, the norm must have the following properties:

- $||v|| \geq 0$
- $||v|| = 0$ iff $v = 0$
- $||k v|| = |k||v||$
- $||v + u|| \leq ||v|| + ||u||$

You can immediately prove the first three properties for the norm defined by

$$||v|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2} = \sqrt{n \sum_{i=1}^{n} (v_i)^2}$$

from the relationship between norm and dot product:

$$||v|| = \sqrt{v \cdot v}$$

(go on, do it)
Other norms

So far, we’ve seen that we can talk about vectors in $\mathbb{R}^n$ ... and vectors in general as anything that satisfies the vector properties. We’ve looked at the dot product specifically on $\mathbb{R}^n$ ... and as an inner product in general as any function that satisfies the inner product properties. So, we have the norm defined as before on $\mathbb{R}^n$ ... and, a norm is any function that satisfies the norm properties. Here they are again, just to remind you:

In general, a **norm** is any function defined on a vector space $V$ that takes a vector as input and returns a scalar as output:

$$ f : V \ni v \rightarrow k \in \mathbb{R} $$

Instead of writing $f(v)$, we generally denote the norm by $||v||$. Furthermore, the norm must have the following properties:

- $||v|| \geq 0$
- $||v|| = 0$ iff $v = 0$
- $||kv|| = |k||v||$
- $||v + u|| \leq ||v|| + ||u||$

When we talked about a general inner product, we looked at an example using continuous functions (so we left the space of vectors in $\mathbb{R}^n$ entirely). To talk about other examples of norms in general, we don’t even have to leave the world of pointy things - there are several functions we can define that take a vector in $\mathbb{R}^n$, return a scalar, and satisfy the properties. The way to verify that a function is a norm is to simply go down the list of properties.

Consider the following function on $\mathbb{R}^n$

$$ ||v||_\infty := \max_{1 \leq i \leq n} |v_i| $$

**Example:** For $v = \langle 1, -7, 6, -2 \rangle$, what is the value of $||v||_\infty$?

Does the function satisfy the properties of a norm?
(more space for those four properties...)
Consider the following function on $\mathbb{R}^n$

$$||v||_1 := \sum_{i=1}^{n} |v_i| = |v_1| + |v_2| + ... + |v_n|$$

**Example:** For $v = <1, -7, 6, -2>$, what is the value of $||v||_1$?

Does the function satisfy the properties of a norm? That’s an assignment question.

**Summary:**

The subscripts outside the $||| \ |||$ are used to distinguish between the different norms on $\mathbb{R}^n$:

- $||v||_\infty := \max_{1 \leq i \leq n} |v_i|$ is called the **max norm** or the **infinity norm**
- $||v||_1 := \sum_{i=1}^{n} |v_i| = |v_1| + |v_2| + ... + |v_n|$ is called the **one norm**
- And so, if we need to distinguish it from the others, we use

$$||v||_2 = \sqrt{v_1^2 + v_2^2 + ... + v_n^2} = \sqrt{\sum_{i=1}^{n} (v_i)^2}$$

for our familiar, Pythagorean theorem, magnitude norm. It is called the **Euclidean norm**.

There is a pattern here, and an infinite number of norms can be defined on $\mathbb{R}^n$. How about a “3 norm”?

$$||v||_3 = \sqrt[3]{|v_1|^3 + |v_2|^3 + ... + |v_n|^3}$$

“A 4 norm”?

$$||v||_4 = \sqrt[4]{v_1^4 + v_2^4 + ... + v_n^4}$$

If the norm is not specified (i.e. the problem just states “find the norm”), assume we’re using the usual Euclidean norm and not some other norm.