Vector spaces

Subspaces

We know that for any \( n \), \( \mathbb{R}^n \) forms a vector space with the operations of vector addition and scalar multiplication. This means that the entire collection of \( n \) dimensional vectors satisfies all the properties listed on p. 208: closure, plus the various addition and scalar multiplication axioms.

The next question is what happens if we consider a subset of an entire collection of vectors, for example all the vectors of the form \( <x, y, 0> \), which is a subset of \( \mathbb{R}^3 \). Do these vectors form a vector space as well? Do they satisfy the 10 properties?

Fortunately, we don’t have to check all the properties. In fact, we don’t have to check any of the vector or scalar multiplication properties. Why not?

The only thing we need to verify is closure. Is the set of vectors of the form \( <x, y, 0> \) closed under vector addition and scalar multiplication?

So, this collection of vectors is a subspace of \( \mathbb{R}^3 \).

**Example:** Does the set of vectors of the form \( <0, y, 1, w> \) form a subspace of \( \mathbb{R}^4 \)?
Example: Does the set of vectors of the form \(<a, b, a + 2b>\) form a subspace of \(\mathbb{R}^3\)?

**Geometric interpretation**

In \(\mathbb{R}^2\) and \(\mathbb{R}^3\), we can interpret subspaces geometrically. What does the set of all vectors of the form \(<a, 2a>\) look like?

Example: What does the set of all vectors of the form \(<a, -a>\) look like?

What does the set of all vectors of the form \(<a, b, a + 2b>\) look like?
The idea of subspaces isn’t restricted to $\mathbb{R}^n$ – we can talk about subspaces of any vector spaces, such as vector spaces of matrices or functions.

**Example:** Does the set of upper triangular matrices in $M_{22}$ form a subspace?

**Example:** Does the set of non-singular matrices in $M_{33}$ form a subspace?
There’s a fast way to check for subspaces if you go a little further down the list of properties. Any subspace of \( \mathbb{R}^n \) must contain the additive identity or zero vector. We can eliminate sets that are not subspaces at a glance if we note that it’s impossible to have a zero. If we do have the zero, this does not imply we’ve got a subspace, however; we still have to check the closure properties.

**Examples:** (Sets which lack zeros)

Is it possible for a set to pass the closure properties, and fail on the additive identity?

So, checking closure is necessary and sufficient to determine whether or not a set is a subspace. The identity property is necessary, but not sufficient: if a set lacks the zero vector, we know it fails the subspace test. If it has a zero, we still need to check closure.