Suggested problems - solutions

Computing the cross product

P1: Let $u = \langle -2, 5, 10 \rangle$, $v = \langle -6, -3, 5 \rangle$. Compute $u \times v$. Verify that $u \times v$ is orthogonal to both $u$ and $v$.

\[
\begin{align*}
  u \times v &= \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle \\
  &= \langle (5) - (10)(-3), -((-2)(5) - (10)(-6)), (-2)(-3) - (5)(-6) \rangle \\
  &= \langle 25 + 30, -(-10 + 60), 6 + 30 \rangle \\
  &= \langle 55, -50, 36 \rangle
\end{align*}
\]

Verify orthogonal to $u$: does $(u \times v) \cdot u = 0$?

\[
\langle 55, -50, 36 \rangle \cdot \langle -2, 5, 10 \rangle = -110 - 250 + 360 = 0
\]

Verify orthogonal to $v$: does $(u \times v) \cdot v = 0$?

\[
\langle 55, -50, 36 \rangle \cdot \langle -6, -3, 5 \rangle = -330 + 150 + 180 = 0
\]

P2: Let $u = i$, $v = j + 2k$. Compute $u \times v$.

\[
\begin{align*}
  u &= \langle 1, 0, 0 \rangle, \ v &= \langle 0, 1, 2 \rangle. \\
  u \times v &= \langle u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1 \rangle \\
  &= \langle (0)(2) - (0)(1), -((1)(2) - (0)(0)), (1)(1) - (0)(0) \rangle \\
  &= \langle 0, -2, 1 \rangle
\end{align*}
\]

P3: Let $u = 4i - 3k$, $v = -12i + 9k$. Compute $u \times v$.

\[
\begin{align*}
  u &= \langle 4, 0, -3 \rangle, \ v &= \langle -12, 0, 9 \rangle. \text{ You can work this one out using the formula, but the fast way is to recall the definition of the cross product as $(||u|| ||v|| \sin \theta)n$. Since it depends on $\sin \theta$, the magnitude of the cross product must be zero when the vectors are parallel.}
\end{align*}
\]

And $u$ and $v$ are parallel; i.e. they are scalar multiples of each other: $v = -3u$.

However, be sure to keep in mind the cross product is always a vector; the answer isn’t 0, but $< 0, 0, 0 >$.

\[
\langle 4, 0, -3 \rangle \times \langle -12, 0, 9 \rangle = < 0, 0, 0 >
\]
P4: Let $\mathbf{u} = <0, 0, -5>$, $\mathbf{v} = <1, -2, 3>$. Find a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Give a unit vector which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ as well.

The cross product by definition gives a third vector orthogonal to both; this question is a restatement of “find $\mathbf{u} \times \mathbf{v}$”.

$$\mathbf{u} \times \mathbf{v} = < u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 >$$

$$= < (0)(3) - (-5)(-2), -(0)(3) - (-5)(1), (0)(-2) - (0)(1) >$$

$$= < -10, -5, 0 >$$

You can check that the vector $<-10, -5, 0>$ is orthogonal to both $\mathbf{u} = <0, 0, -5>$ and $\mathbf{v} = <1, -2, 3>$ by dotting.

Then, to find a unit vector, normalize your answer:

$$\begin{align*}
\frac{1}{\sqrt{125}} < -10, -5, 0 > &= \frac{1}{\sqrt{5}} < -2, -1, 0 > \\
\end{align*}$$

is still orthogonal to both.

P5: Let $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Find a vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Give a unit vector which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ as well.

Orthogonal vector:

$$\mathbf{u} \times \mathbf{v} = < -(4)(-2) - (1)(1), -(3)(-2) - (1)(1), (3)(1) - (-4)(1) > = < 7, 7, 7 >$$

Orthogonal unit vector:

$$\begin{align*}
\frac{1}{\sqrt{147}} < 7, 7, 7 > &= < \frac{1}{3\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{3\sqrt{3}} > \\
\end{align*}$$

Additional problems of this type at COW: