The cross product

Defining the cross product - suggested problems - solutions

P1: Suppose we have two vectors $\mathbf{u}$ and $\mathbf{v}$ in space, such that $||\mathbf{u}|| = 25$, $||\mathbf{v}|| = 17$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $110^\circ$. What is the magnitude of the cross product? (Compute $||\mathbf{u} \times \mathbf{v}||$.) Why is it not possible to determine the direction, based on the given information?

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| ||\mathbf{v}|| \sin \theta = (25)(17) \sin 110^\circ \approx 399.4$$

We cannot determine direction (in any useful way - we know the direction is “perpendicular to both”) as we have no idea where $\mathbf{u}$ and $\mathbf{v}$ are pointed in space, so there is no way to apply the right hand rule.

P2: Compute the cross products $\mathbf{j} \times \mathbf{k}$ and $\mathbf{k} \times \mathbf{i}$. In each case, be sure to sketch the vectors on a right handed 3D coordinate system, and use the right hand rule to determine the correct direction.

$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$

Fingers along $\mathbf{j}$, palm up so you can curl towards $\mathbf{k}$, and your thumb will be sticking out along the positive $x$ axis, giving the unit vector $\mathbf{i}$. 
**k \times i = j**

Fingers along \( k \), palm forward so you can curl towards \( i \) out of the page, and your thumb will be pointed along the positive \( y \) axis, giving the unit vector \( j \).

Notice how if it had been \( i \times k \), you’d have had to turn your hand the other way, and would get \(-j\). Reversing the order of the cross product reverses the sign.

**P3:** Let \( u = 3i - 2j + 0k \), and \( v = i + 4j + 0k \).

(a) Note both vectors lie in the \( xy \) plane; sketch them.

(b) Compute \( ||u||, ||v||, \) and \( u \cdot v \).

\[
||u|| = \sqrt{13} \quad ||v|| = \sqrt{17} \quad u \cdot v = (3)(1) + (-2)(4) + (0)(0) = -5
\]

(c) Compute the angle between \( u \) and \( v \) from the dot product formula: \( \cos \theta = \frac{u \cdot v}{||u|| ||v||} \).

\[
\cos \theta = \frac{-5}{\sqrt{13}\sqrt{17}} \quad \theta \approx 109.65^\circ
\]
(d) Use the preceding to compute $||u \times v||$. What is $||v \times u||$?

$$||u \times v|| = ||u|| \cdot ||v|| \sin \theta = \sqrt{13} \sqrt{17} \sin 109.65^\circ = 14$$

A point of interest here is that it works out to be exactly 14; if instead of calculating $\theta$ as a decimal approximation, you use the trig identity $\sin^2 \theta = 1 - \cos^2 \theta$, you have

$$||u|| \cdot ||v|| \sin \theta = \sqrt{13} \sqrt{17} \sqrt{1 - \left( \frac{-5}{\sqrt{13\sqrt{17}}} \right)^2}$$

$$= \sqrt{13} \sqrt{17} \sqrt{1 - \frac{25}{(13)(17)}}$$

$$= \sqrt{221} \sqrt{1 - \frac{196}{221}}$$

$$= \sqrt{196}$$

$$= 14$$

This is just generally more trouble than it’s worth :) 

$$||v \times u|| = ||v|| \cdot ||u|| \sin \theta = ||u|| \cdot ||v|| \sin \theta = ||u \times v||$$

(they’re the same), and so $||v \times u|| = 14$ as well.

(e) Use your sketch and the right hand rule to determine the direction of $u \times v$. Does your thumb go into or out of the plane of the page? In a right handed coordinate system, “into” points down the negative z axis, and the unit direction vector $\mathbf{n}$ is equal to $-\mathbf{k}$. “Out of” points up the positive z axis, and the unit direction vector $\mathbf{n}$ is equal to $\mathbf{k}$.

Fingers along $\mathbf{u}$ curling towards $\mathbf{v}$ requires you to hold your hand with thumb pointing out of the plane of the page, so the direction vector is $\mathbf{k}$. One convention for indicating an “out” vector in a 2D sketch is to use a circle with a dot (like the pointy tip of an arrow coming out towards you):
(f) Write the vector \( \mathbf{u} \times \mathbf{v} \).

The vector \( \mathbf{u} \times \mathbf{v} \) is given by \( (||\mathbf{u}|| ||\mathbf{v}|| \sin \theta)\mathbf{n} \): magnitude times direction. So

\[
\mathbf{u} \times \mathbf{v} = 14\mathbf{k} = <0, 0, 14>
\]

(g) Use your sketch and the right hand rule to determine the direction of \( \mathbf{v} \times \mathbf{u} \). Does your thumb go into or out of the plane of the page?

Fingers along \( \mathbf{v} \) curling towards \( \mathbf{u} \) requires you to hold your hand with thumb pointing into the page, so the direction vector is \( -\mathbf{k} \). One convention for indicating an “in” vector in a 2D sketch is to use a circle with a \( \times \) (like the tail end of an arrow moving away from you):

(h) Write the vector \( \mathbf{v} \times \mathbf{u} \).

\[
\mathbf{v} \times \mathbf{u} = ||\mathbf{v}|| ||\mathbf{u}|| \sin \theta \mathbf{n} = 14(-\mathbf{k}) = -14\mathbf{k} = <0, 0, -14>
\]
P4: Let $\mathbf{u} = <-1, 1, 0>$, $\mathbf{v} = <2, 3, 0>$. Find $\mathbf{u} \times \mathbf{v}$.

There's no shortcut to this yet; you need to go through all the steps in problem 3 to find that cross product. I'm just not listing them all out again! We will go through another formula to get the cross product; even if you know it already, take the geometric approach as above.

$$||\mathbf{u}|| = \sqrt{2} \quad ||\mathbf{v}|| = \sqrt{13} \quad \mathbf{u} \cdot \mathbf{v} = 1$$

$$\theta = \cos^{-1} \frac{1}{\sqrt{2}\sqrt{13}} \approx 78.69^\circ$$

$$||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}|| ||\mathbf{v}|| \sin 78.69^\circ = 5$$

Direction by right hand rule is into the page, so $\mathbf{n} = -\mathbf{k}$.

$$\mathbf{u} \times \mathbf{v} = ||\mathbf{u} \times \mathbf{v}|| \mathbf{n} = -5\mathbf{k} = <0, 0, -5>$$