Lines and planes

Planes - suggested problems - solutions

P1: A plane passing through the point \((1, 4, 7)\) has normal vector \(\mathbf{n} = \langle 9, 4, -1 \rangle\). Write an equation for the plane in point-normal form. Transform this equation into scalar form, then general form.

Point-normal \((\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0)\):

\[
\langle 9, 4, -1 \rangle \cdot \langle x - 1, y - 4, z - 7 \rangle = 0
\]

Scalar:

\[
9(x - 1) + 4(y - 4) - (z - 7) = 0
\]

General:

\[
9x - 9 + 4y - 16 - z + 7 = 0
\]

\[
9x + 4y - z = 18
\]

Additional problems of this type at COW:

3. Calculus Book III
   > 2. Vectors and Analytic Geometry
   > 5. Lines and Planes
   > 2. Equations of planes I

Equations should be transformed into general form for entering in COW.
P2: A plane has the equation $3x - 2y + z = 10$.

(a) Find a normal vector for the plane.

$$\mathbf{n} = <3, -2, 1>$$

(b) Find a unit normal vector for the plane. How many of these are there?

$$||\mathbf{n}|| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$$

There are two unit normals (both of length 1, but in opposite directions):

$$\frac{1}{\sqrt{14}} <3, -2, 1> \quad \text{and} \quad -\frac{1}{\sqrt{14}} <3, -2, 1>$$

(c) Find the $x$, $y$, and $z$ intercepts of the plane.

$x$ intercept (set $y$ and $z$ equal to zero): $3x = 10$, so $x = \frac{10}{3}$. Point is $(\frac{10}{3}, 0, 0)$.

$y$ intercept (set $x$ and $z$ equal to zero): $-2y = 10$, so $y = -5$. Point is $(0, -5, 0)$.

$z$ intercept (set $x$ and $y$ equal to zero): $z = 10$. Point is $(0, 0, 10)$.

(d) Hand sketch the plane.

Hand sketches of planes are done from the intercepts, and the convention is to show a triangular slice (the plane obviously extends beyond that):

(e) Write the equation in $z = f(x, y)$ form.

$$3x - 2y + z = 10$$

$$z = -3x + 2y + 10$$

$$f(x, y) = -3x + 2y + 10$$
P3: Find the angle between the planes $2x - 3y = 7$ and $-x + y + z = 4$.

The angle between planes is the angle between their normal vectors. Keeping in mind that normals can point either way, we generally shoot for the acute angle, so put an absolute value in the dot product below. The planes above have $\mathbf{n}_1 = \langle 2, -3, 0 \rangle$ and $\mathbf{n}_2 = \langle -1, 1, 1 \rangle$.

The angle between two vectors comes from the dot product formula $\cos \theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{||\mathbf{u}|| ||\mathbf{v}||}$, so

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{||\mathbf{n}_1|| ||\mathbf{n}_2||}$$
$$= \frac{|2(-1) + (-3)(1) + 0(1)|}{\sqrt{2^2 + (-3)^2 + 0^2} \sqrt{(-1)^2 + 1^2 + 1^2}}$$
$$= \frac{|-5|}{\sqrt{13}\sqrt{3}}$$
$$\theta = \cos^{-1} \left( \frac{5}{\sqrt{39}} \right)$$
$$\theta \approx .642 \text{ radians or } 36.81^\circ$$

P4: Determine whether the planes are parallel, orthogonal, or neither. If neither, give the angle between the planes.

Use the angle formula as above and interpret the results:

(a) $5x - 3y + z = 4$ and $x + 4y + 7z = 1$

$\mathbf{n}_1 = \langle 5, -3, 1 \rangle$, $\mathbf{n}_2 = \langle 1, 4, 7 \rangle$, and

$$\cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{||\mathbf{n}_1|| ||\mathbf{n}_2||}$$
$$= \frac{|5(1) + (-3)(4) + 1(7)|}{\sqrt{5^2 + (-3)^2 + 1^2} \sqrt{1^2 + 4^2 + 7^2}}$$
$$= \frac{0}{\sqrt{35}\sqrt{66}}$$
$$\theta = \cos^{-1} 0$$
$$\theta = 90^\circ$$

Orthogonal.
(b) $x - 3y + 6z = 4$ and $5x + y - z = 4$

$\mathbf{n_1} = < 1, -3, 6 >, \mathbf{n_2} = < 5, 1, -1 >$, and

$$
\cos \theta = \frac{|\mathbf{n_1} \cdot \mathbf{n_2}|}{||\mathbf{n_1}|| \cdot ||\mathbf{n_2}||} = \frac{|1(5) + (-3)(1) + 6(-1)|}{\sqrt{1^2 + (-3)^2 + 6^2} \cdot \sqrt{5^2 + 1^2 + (-1)^2}} = \frac{|-4|}{\sqrt{46} \sqrt{27}}
$$

$$
\theta = \cos^{-1} \left( \frac{4}{\sqrt{1242}} \right)
$$

$$
\theta \approx 1.457 \text{ radians or } 83.48^\circ
$$

Neither orthogonal nor parallel.

(c) $x - 5y - z = 1$ and $5x - 25y - 5z = -3$ $\mathbf{n_1} = < 1, -5, -1 >, \mathbf{n_2} = < 5, -25, -5 >$. It’s already obvious that the normals are multiples of each other, but you’ll also see it if you go for the angle formula;

$$
\cos \theta = \frac{|\mathbf{n_1} \cdot \mathbf{n_2}|}{||\mathbf{n_1}|| \cdot ||\mathbf{n_2}||} = \frac{|1(5) + (-5)(-25) + (-1)(-5)|}{\sqrt{1^2 + (-5)^2 + (-1)^2} \cdot \sqrt{5^2 + (-25)^2 + (-5)^2}} = \frac{|135|}{\sqrt{27} \sqrt{675}}
$$

$$
\theta = \cos^{-1} \left( \frac{135}{\sqrt{18225}} \right) = \cos^{-1} 1
$$

$$
\theta = 0^\circ
$$

The planes are parallel...or possibly the same plane. To rule that out, notice they do not intersect by trying to solve simultaneously:

$$
\begin{align*}
-5(x - 5y - z) &= 1 \\
5x - 25y - 5z &= -3 \\
0 &= -8
\end{align*}
$$